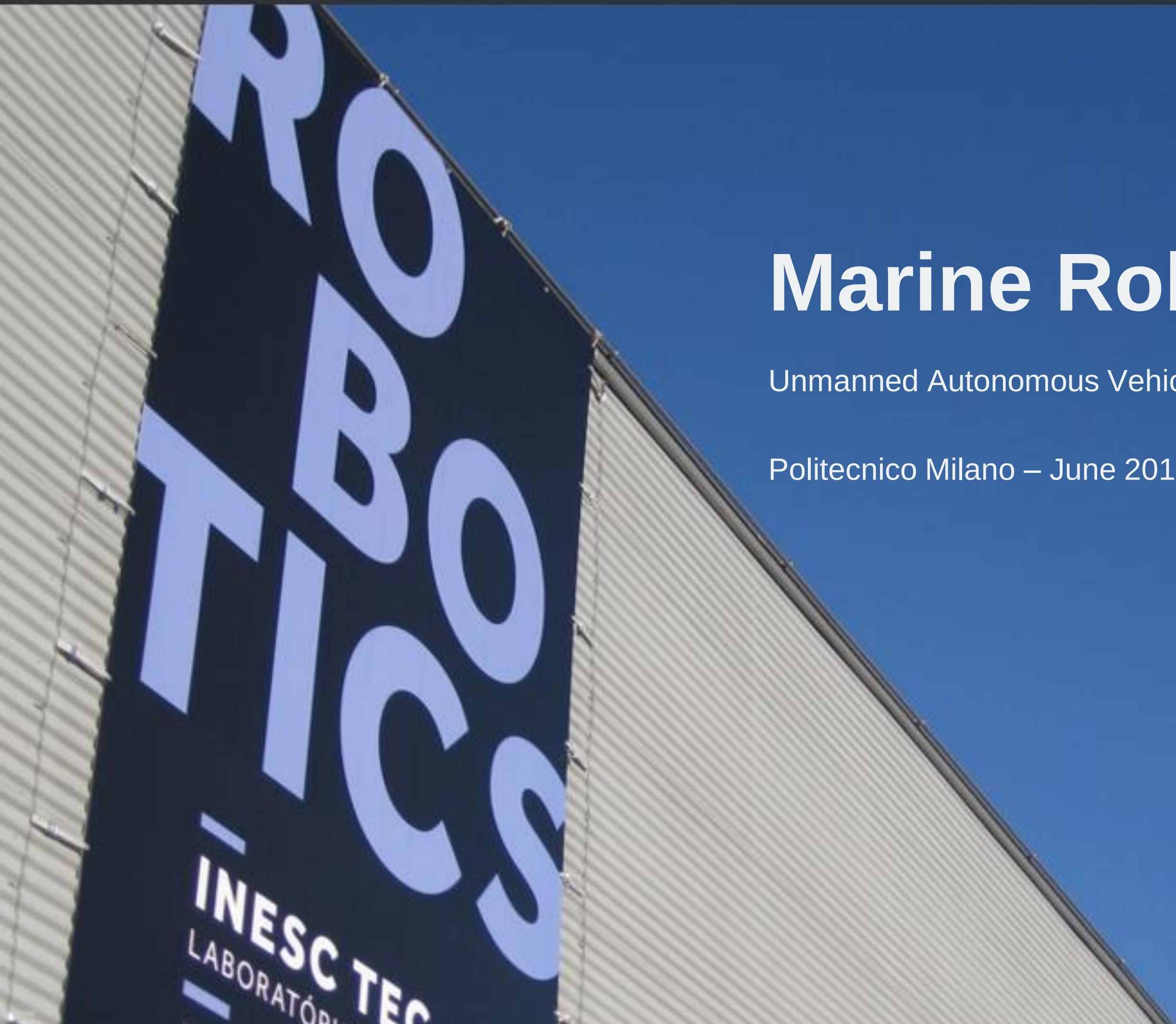
Marine Robotics

Unmanned Autonomous Vehicles in Air Land and Sea

Politecnico Milano – June 2016



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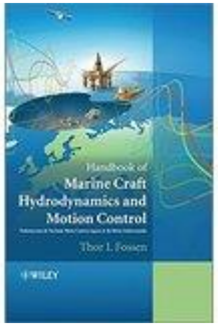
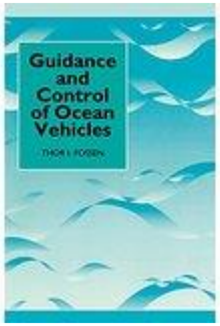
Marine craft modeling

References

- Thor Fossen's books
  - Handbook of Marine Craft Hydrodynamics and Motion Control
  - Guidance and Control of Ocean Vehicles
- J. N. Newman, Marine Hydrodynamics
- S.F. Hoerner : Fluid Dynamic Drag (1965), Fluid Dynamic Lift (1992)





- Courses
  - MIT OCW 2.154/13.49 - Maneuvering and control of surface and underwater vehicles

- Some useful Matlab Toolboxes
  - Marine Systems Simulator (MSS) (www.marinecontrol.org)
  - Robotics and Machine Vision Toolboxes (www.petercorke.com)

Marine craft

- Ships, floating rigs, submarines, ROVs, AUVs, etc
- Hydrodynamics determined by Froude number
  - Hydrostatic pressure (buoyancy)
  - Hydrodynamic pressure (drag, lift)

$$Fn := \frac{U}{\sqrt{gL}}$$

U- vehicle speed
g- gravity acceleration
L- vehicle lenght

- At constant speed U
  - **Displacement vessels (Fn<0.4)**
  - Semi-displacement vessel (0.4 < Fn <1.0)
  - Planning vessel (Fn>1.0)

[1] Faltinsen, O.M., "Hydrodynamic of High-Speed Marine Vehicles" Cambridge University Press, 2005

Naval architecture classical models

- **Maneuvering theory**
  - Constant speed in calm water
  - Hydrodynamic coefficients frequency independent (no wave excitation)
  - Hydrodynamic forces usually represented by Taylor expansion (hydrodynamic coefficients)

- Seakeeping theory
  - Hydrodynamic coefficients, wave forces computed as function of wave excitation frequency (using hull geometry and mass distribution)

Model usage

- Simulation
  - Accurate description
  - Usually full 6DOF
  - For simulation and control validation
- Control design
  - Simplified models (linearized, etc)
  - Decoupled motion models
  - Possibly fewer DOFs
- Observer design
  - Modeling sensor and navigation dynamics
  - Disturbance modeling and estimation (wave, wind, currents often modeled as colored noise)

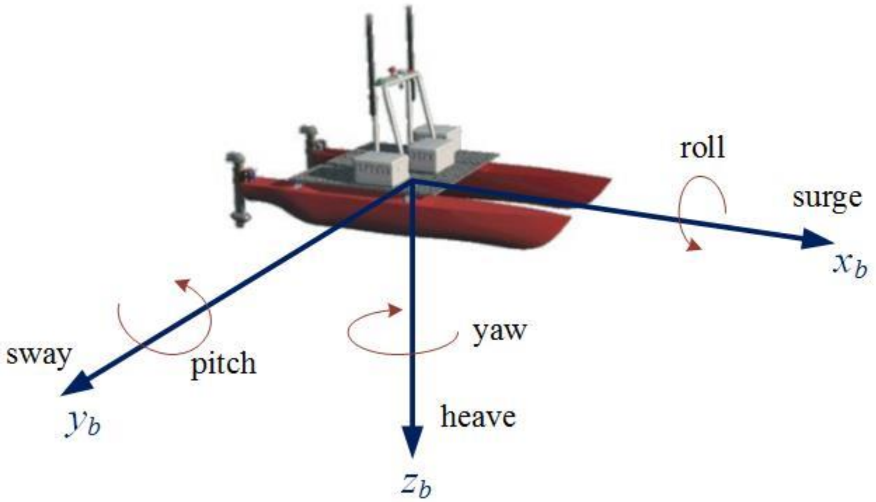
DOFs

- 1 DOF
  - Design of forward speed controllers, heading controllers or roll damping controllers

- 3 DOF
  - **Horizontal plane models** (surge, sway and yaw) for positioning and path following
  - **Longitudinal models** (surge, heave and pitch) for vertical motion control
  - Lateral models (sway, roll and yaw) for turning and heading control (mainly for ships)

- 4DOF
  - (surge, sway, roll and yaw) added roll to horizontal model for roll stabilization maneuvers

- 6DOF
  - **Full 6 DOF** motion used for simulation and advanced control actuated in all degrees



Earth centered reference frames

- ECI – Earth centered inertial
  - Inertial frame non accelerating
  - Newton laws apply
  - Does not rotate with Earth

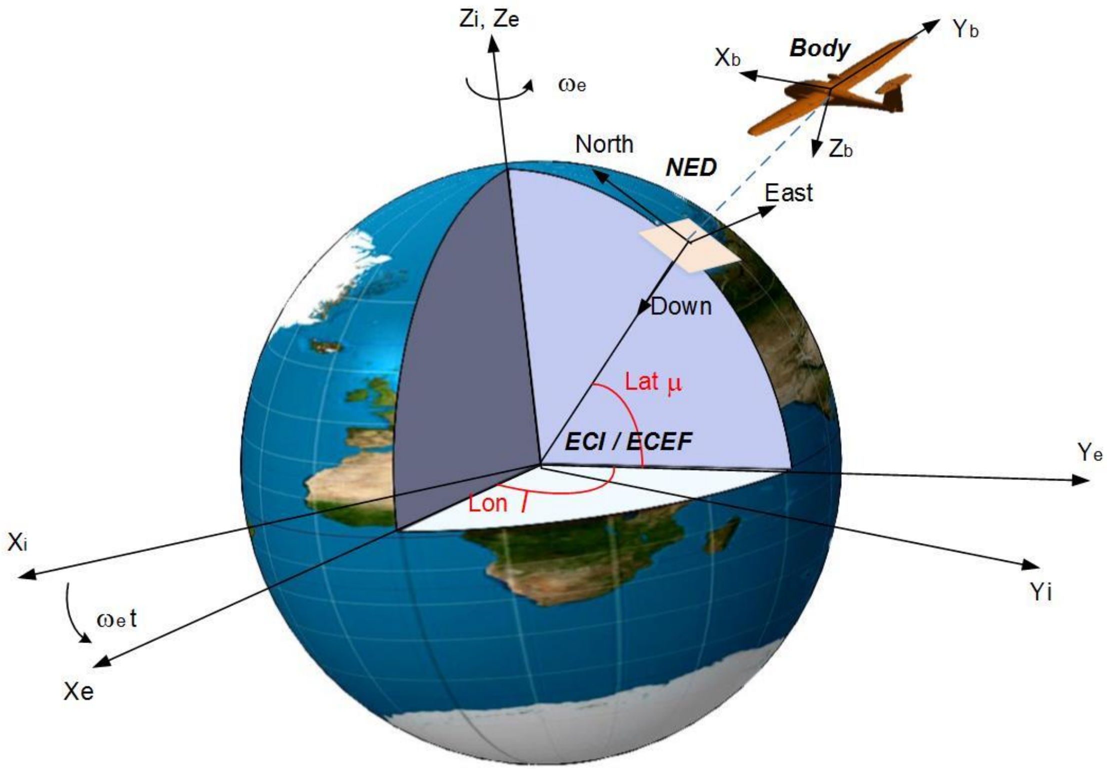
- ECEF – Earth centered Earth fixed
  - Origin fixed in Earth center
  - Axes rotating with Earth rotation (relatively to ECI (at rate 7.2921x10-5 rad/s)

Geographic reference frames

- NED – North-East-Down
  - Relative to Earth reference ellipsoid (WGS-84)
  - Defined as tangent plane o surface moving with vehicle - for local navigation (approx**. constant lat. lon.**) can be considered fixed on surface

Reference frames

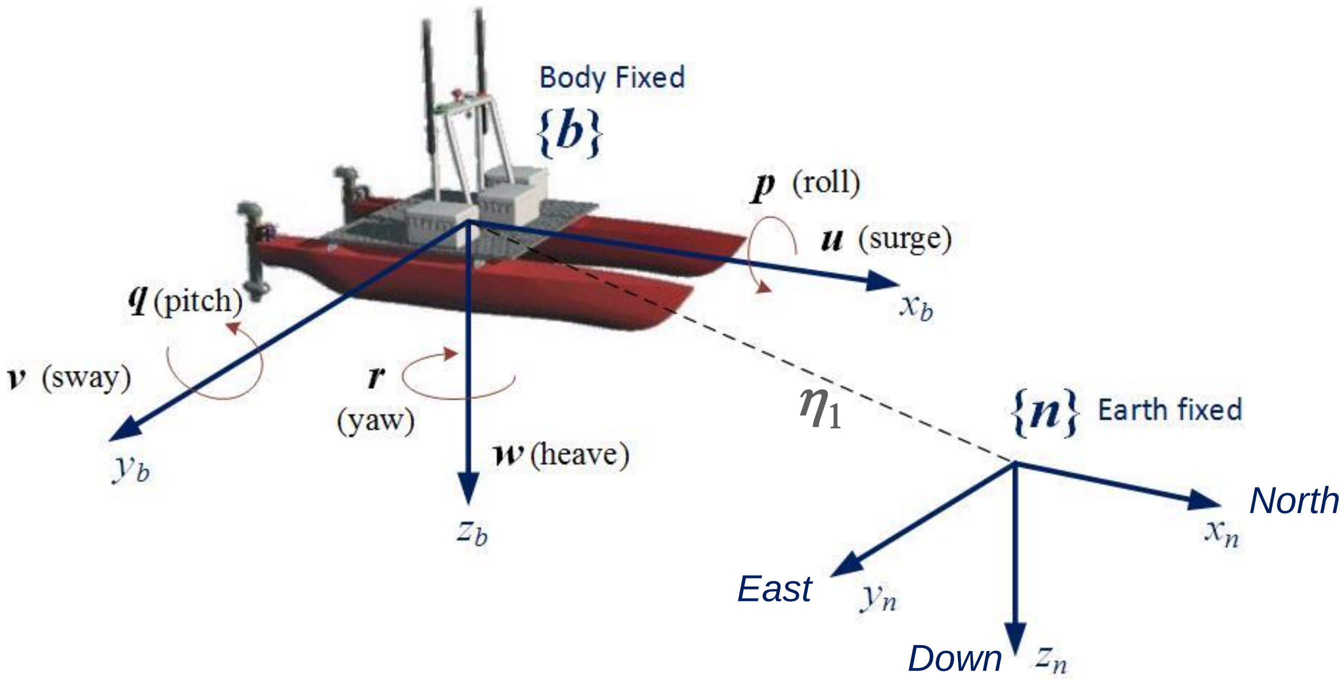
- ECI { i } –Earth Centered Inertial (fixed in space)
- ECEF {e} – Earth Centered Earth Fixed –(rotating with ang. vel ωe)
- NED {n}– North-East-Down (local)
- ENU – East-North-Up (local)
- BODY {b} – Body fixed



Maneuvering reference frames

- Considering motion of marine vehicles, 2 reference frames are usually used for 6DOF maneuvering models:
  - **{n}** ***Local inertial NED frame*** – navigation or world frame, for local motion can be considered as fixed at the surface (**flat earth navigation**) and **inertial** (neglecting earth accel.), defines a tangent plane to surface with axes pointing North, East and Down
  - **{b}** ***Body frame*** – fixed with the vehicle and aligned with the vehicle principal symmetry axis, (body velocities usually expressed in this frame)

Reference frames



Notation from SNAME [1]

	Forces and Moments	Body-Fixed Velocities	Inertial Position and Euler Angles
motions in the x-direction (surge)	X	u	x
motions in the y-direction (sway)	Y	v	y
motions in the z-direction (heave)	Z	w	z
rotation about the x-axis (roll)	K	p	ϕ
rotation about the y-axis (pitch)	M	q	θ
rotation about the z-axis (yaw)	N	r	ψ

[1] "Nomenclature for Treating the Motion of a Submerged Body Through a Fluid" The Society of Naval Architects and Marine Engineers, Technical and Research Bulletin No.1--5, April 1950, pp.1--15

Course and sideslip

- Horizontal motion, course and sideslip angles

Heading: ψ

Course angle: χ

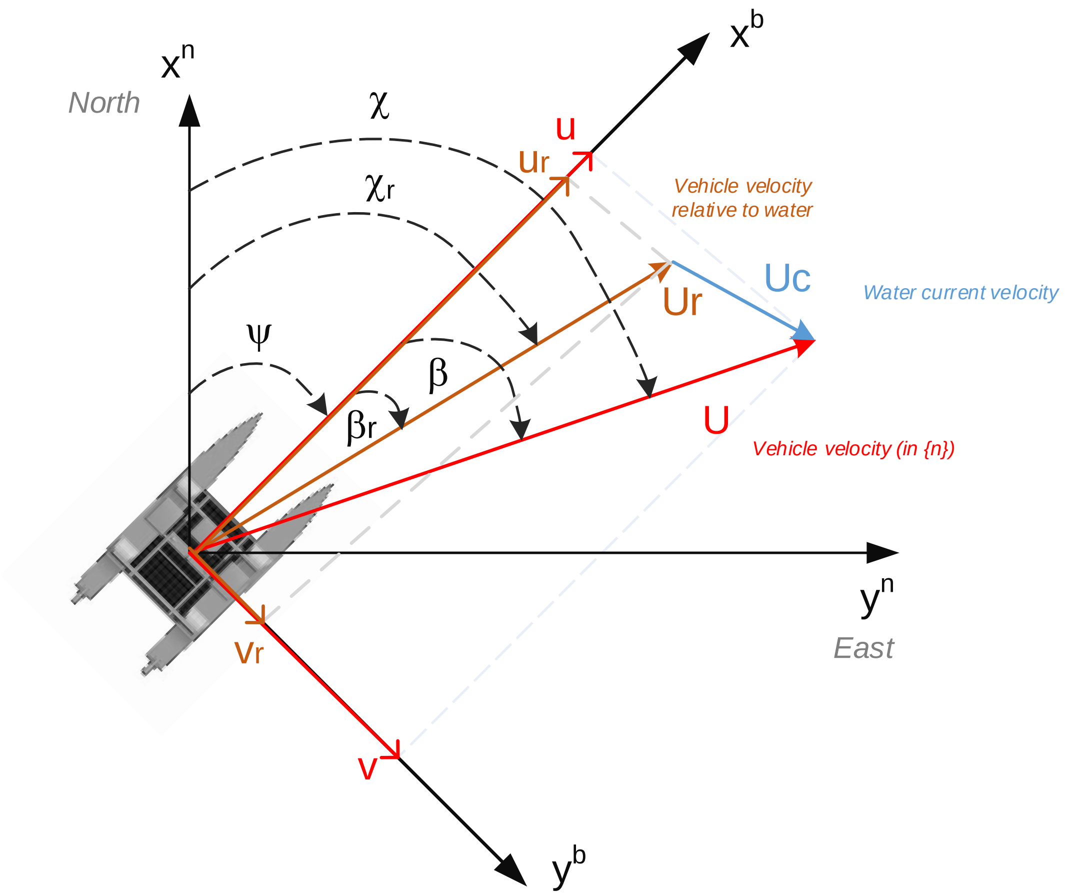
Sideslip angle: $\beta_r = \arcsin(\frac{v_r}{U_r})$

Crab angle: β

Vehicle velocity: $U = \sqrt{u^2 + v^2}$

Vehicle relative velocity (to the water): $U_r = \sqrt{u_r^2 + v_r^2}$

Water current velocity: U_c



Angle of attack

- Vertical motion, angle of attack

Pitch angle: θ

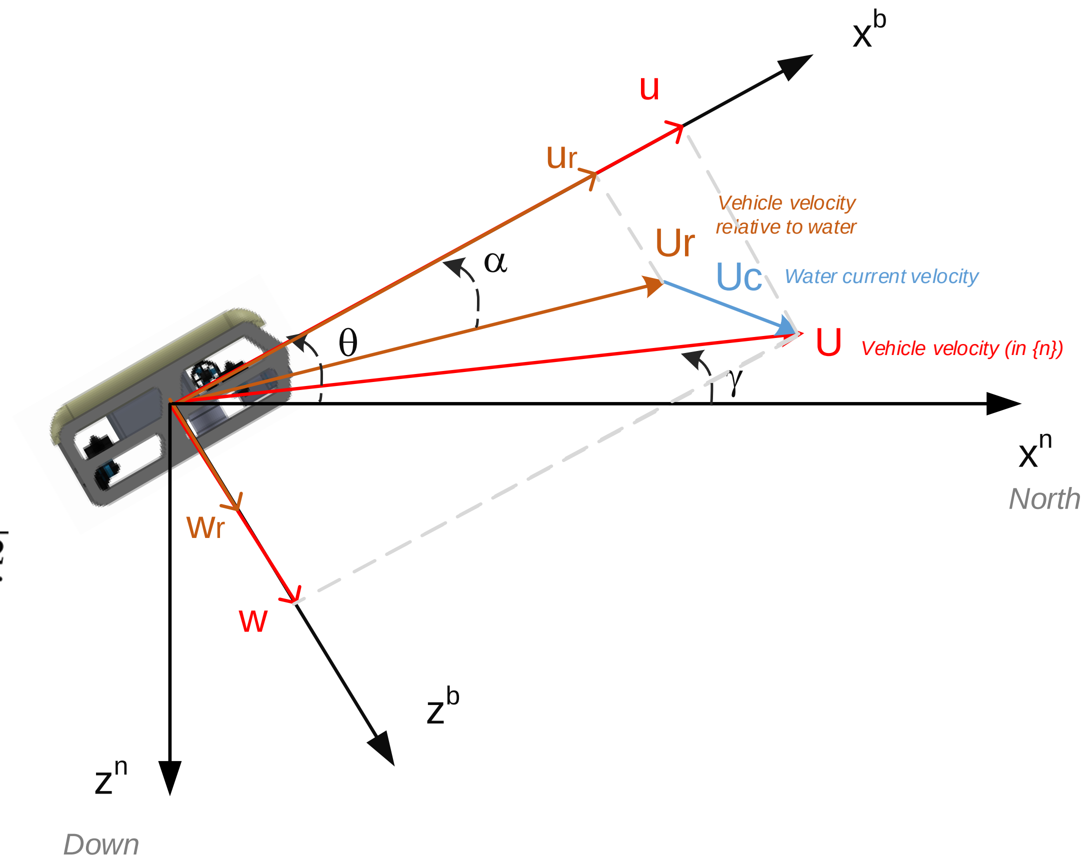
Flight path angle: λ

Angle of attack: $\alpha_r = \arctan(\frac{w_r}{u_r})$

Vehicle velocity: $U = \sqrt{u^2 + w^2}$

Vehicle relative velocity (to the water): $U_r = \sqrt{u_r^2 + w_r^2}$

Water current velocity: U_c



Notation

- Position and orientation expressed in **{n}**
- Velocities and generalized forces (forces and moments) expressed in **{b}**

$$\eta_1 = [\ x \quad y \quad z\ ]^T; \qquad \eta_2 = [\ \phi \quad \theta \quad \psi\ ]^T$$

$$\nu_1 = [\ u \quad v \quad w\ ]^T; \qquad \nu_2 = [\ p \quad q \quad r\ ]^T$$

$$\tau_1 = [\ X \quad Y \quad Z\ ]^T; \qquad \tau_2 = [\ K \quad M \quad N\ ]^T$$

$$\boldsymbol{\eta} = \begin{bmatrix} \boldsymbol{\eta}_1 \\ \boldsymbol{\eta}_2 \end{bmatrix} \qquad \boldsymbol{\nu} = \begin{bmatrix} \boldsymbol{\nu}_1 \\ \boldsymbol{\nu}_2 \end{bmatrix} \qquad \boldsymbol{\tau} = \begin{bmatrix} \boldsymbol{\tau}_1 \\ \boldsymbol{\tau}_2 \end{bmatrix}$$

$\boldsymbol{\eta}$	earth fixed position and attitude vector (position and orientation of the vehicle with respect to the **{n}** frame)
$\boldsymbol{\nu}$	body fixed linear and angular velocity (velocities of the vehicle with respect to the **{b}** frame)
$\boldsymbol{\tau}$	forces and moments acting on the vehicle with respect to the **{b}** frame

Transformation between {n} and {b}

- Translation of {b} in relation to {n}

$$\eta_1 = \begin{bmatrix} x & y & z \end{bmatrix}^T$$

- {b} can have an arbitrary 3D rotation in relation to {n}
  - rotation between two 3D reference frames can be expressed by a rotation matrix ***R***
  - Multiple parameterizations possible to represent the rotation (Euler angles, quaternions)

Rotation matrices

Orthogonal matrices of order 3

$$O(3) := \left\{ \boldsymbol{R} | \boldsymbol{R} \in \mathbb{R}^{3\times 3}, \quad \boldsymbol{R}\boldsymbol{R}^\top = \boldsymbol{R}^\top \boldsymbol{R} = \boldsymbol{I} \right\}$$

Special orthogonal group of order 3

$$SO(3) = \left\{ \boldsymbol{R} | \boldsymbol{R} \in \mathbb{R}^{3\times 3}, \quad \boldsymbol{R} \text{ is orthogonal and } \det \boldsymbol{R} = 1 \right\}$$

Rotation matrix

$$\boldsymbol{R}\boldsymbol{R}^\top = \boldsymbol{R}^\top \boldsymbol{R} = \boldsymbol{I}, \qquad \det \boldsymbol{R} = 1$$

Inverse rotation

$$\boldsymbol{R}^{-1} = \boldsymbol{R}^\top$$

notation

$$\mathbf{v}^{to} = \mathbf{R}^{to}_{from} \mathbf{v}^{from}$$

Euler's rotation theorem

- Every change in relative orientation between two reference frames **{A}** and **{B}** can be produced by a simple rotation of **{B}** in **{A}** (a rotation about an axis)

- Any two independent orthonormal coordinate frames can be related by a sequence of rotations (not more than three) about coordinate axes, where no two successive rotation may be about the same axis [1]

[1] J.B. Kuipers, "Quaternions and rotation sequences: A primer with applications to orbits, aerospace and virtual reality". Princeton University Press, 1999

Rotation matrices

Rotation matrix $\boldsymbol{R}$ has one real eigenvalue $\lambda = 1$ and a complex pair $\lambda = \cos\theta \pm i\sin\theta$

$$\boldsymbol{Rv} = \lambda\boldsymbol{v}$$

$\boldsymbol{v}$ is the eigenvector corresponding to λ

$$\lambda = 1 \qquad \boldsymbol{Rv} = \boldsymbol{v}$$

so $\boldsymbol{v}$ is unchanged by the rotation thus $\boldsymbol{v}$ is vector about which the rotation of θ occurs

The rotation matrix R can be obtained from an angle and vector using the Rodrigues formula

$$\mathbf{R} = \mathbf{I}_{3\times3} + sin\theta\mathbf{S}(\mathbf{v}) + (1 - cos\theta)(\mathbf{v}\mathbf{v}^T - \mathbf{I}_{3\times3})$$

with $\mathbf{S}()$ is the skew-symmetric matrix

Skew-symmetric matrix

- Cross product can be defined as matrix-vector multiplication

$$\mathbf{v} \times \mathbf{a} := \mathbf{S}(\mathbf{v})\mathbf{a}$$

- with $\mathbf{S}()$ the skew-symmetric matrix defined as (for dimension 3):

$$\mathbf{S}(\mathbf{v}) = -\mathbf{S}^T(\mathbf{v}) = \begin{bmatrix} 0 & -\lambda_3 & \lambda_2 \\ \lambda_3 & 0 & -\lambda_1 \\ -\lambda_2 & \lambda_1 & 0 \end{bmatrix} \qquad \mathbf{v} = \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix}$$

matrix S is anti-symmetric or skew-symmetric if $\mathbf{S} = -\mathbf{S}^T$

Simple rotation about an axis

- Considering $\mathbf{v}^b$ a vector fixed in **{b}** the Euler rotation theorem implies that there exists one axis and an angle such that a simple rotation would rotate it to NED **{n}**, thus

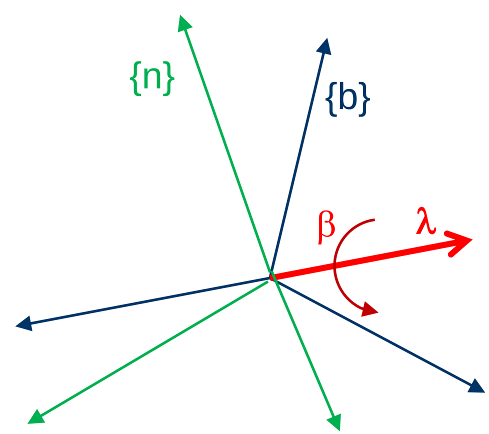
$$\mathbf{v}^n = \mathbf{R}_b^n \mathbf{v}^b \qquad \mathbf{R}_b^n = \mathbf{R}_{\lambda,\beta}$$

- Considering a unit vector $\boldsymbol{\lambda} = [\lambda_1, \lambda_2, \lambda_3]^\top, \quad |\boldsymbol{\lambda}| = 1$ defining the axis of rotation, the rotation matrix describing a rotation of β about this axis is given by

$$\mathbf{R}_{\lambda,\beta} = \mathbf{I}_{3\times3} + sin\beta\mathbf{S}(\lambda) + (1 - cos\beta)(\lambda\lambda^T - \mathbf{I}_{3\times3})$$

- expanding

$$\begin{aligned}
R_{11} &= [1 - \cos(\beta)]\,\lambda_1^2 + \cos(\beta) \\
R_{22} &= [1 - \cos(\beta)]\,\lambda_2^2 + \cos(\beta) \\
R_{33} &= [1 - \cos(\beta)]\,\lambda_3^2 + \cos(\beta) \\
R_{12} &= [1 - \cos(\beta)]\,\lambda_1\lambda_2 - \lambda_3\sin(\beta) \\
R_{21} &= [1 - \cos(\beta)]\,\lambda_2\lambda_1 + \lambda_3\sin(\beta) \\
R_{23} &= [1 - \cos(\beta)]\,\lambda_2\lambda_3 - \lambda_1\sin(\beta) \\
R_{32} &= [1 - \cos(\beta)]\,\lambda_3\lambda_2 + \lambda_1\sin(\beta) \\
R_{31} &= [1 - \cos(\beta)]\,\lambda_3\lambda_1 - \lambda_2\sin(\beta) \\
R_{13} &= [1 - \cos(\beta)]\,\lambda_1\lambda_3 + \lambda_2\sin(\beta)
\end{aligned}$$



Sequence of rotations

- The rotation about an axis can be composed by a sequence of rotations about the coordinate axes
- Rotation matrices about each coordinate axis (principal rotations) can be obtained by setting:

$$\boldsymbol{\lambda} = [0, 0, 1]^{\top} \qquad \beta = \psi$$

$$\boldsymbol{\lambda} = [0, 1, 0]^{\top} \qquad \beta = \theta$$

$$\boldsymbol{\lambda} = [1, 0, 0]^{\top} \qquad \beta = \phi$$

substituting:

$$\mathbf{R}_{z,\psi} = \begin{bmatrix} cos\psi & -sin\psi & 0 \\ sin\psi & cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \mathbf{R}_{y,\theta} = \begin{bmatrix} cos\theta & 0 & sin\theta \\ 0 & 1 & 0 \\ -sin\theta & 0 & cos\theta \end{bmatrix} \quad \mathbf{R}_{x,\phi} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & cos\phi & -sin\phi \\ 0 & sin\phi & cos\phi \end{bmatrix}$$

Sequence of rotations

- The rotation matrix from **{n}** to **{b}** can be given by a sequence of principal rotations using **zyx convention**

$$\mathbf{R}_n^b(\phi,\theta,\psi) = \mathbf{R}_b^n(\phi,\theta,\psi)^T$$

$$\mathbf{v}^n = \mathbf{R}_b^n \mathbf{v}^b$$

Rotation from {b} to {n}

$$\mathbf{R}_b^n(\phi,\theta,\psi) = \mathbf{R}_b^n(\eta_2) := \mathbf{R}_{z,\psi}\mathbf{R}_{y,\theta}\mathbf{R}_{x,\phi}$$

$$\mathbf{R}_n^b(\phi,\theta,\psi) = \mathbf{R}_b^n(\phi,\theta,\psi)^{-1} = \mathbf{R}_{x,\phi}^T\mathbf{R}_{y,\theta}^T\mathbf{R}_{z,\psi}^T$$

The order of rotations it is not arbitrary!

There are 12 possible sequences of rotations (*xyx*, *xzx*, *yxy*, *yzy*, *zxz*, *zyz*, *xyz*, *xzy*, *yzx*, *yxz*, *zxy* and ***zyx***)

Eulerian rot.(one axis repeating) — *xyx, xzx, yxy, yzy, zxz, zyz*

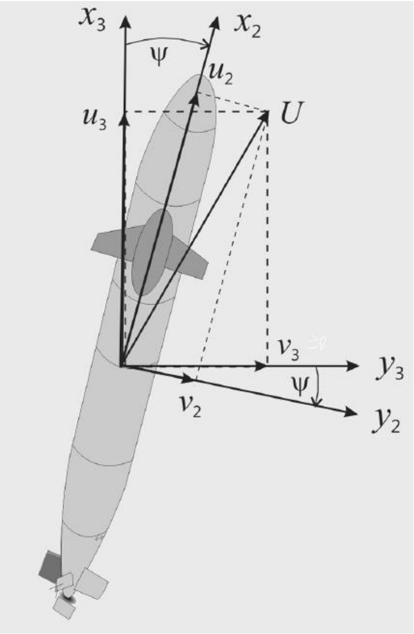
Cardanian rot. — *xyz, xzy, yzx, yxz, zxy, zyx*

Euler angle rotation sequence

Rotation from {n} to {b}

1 - Rotation over yaw angle ψ about z_3
2 - Rotation over pitch angle θ about y_2
3 - Rotation over roll angle ϕ about x_1

1



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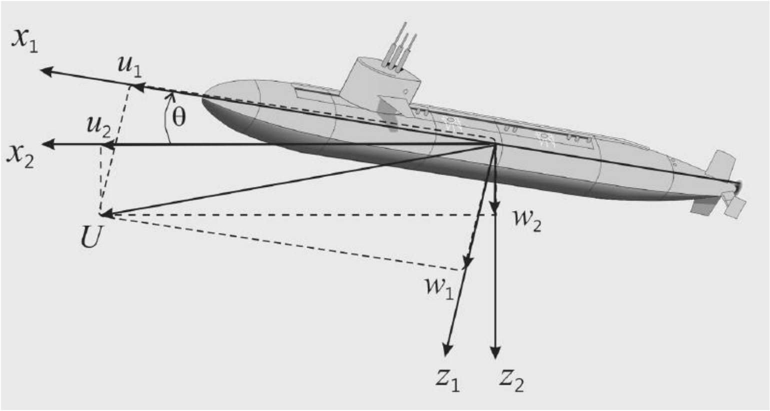

Pictures from [1]

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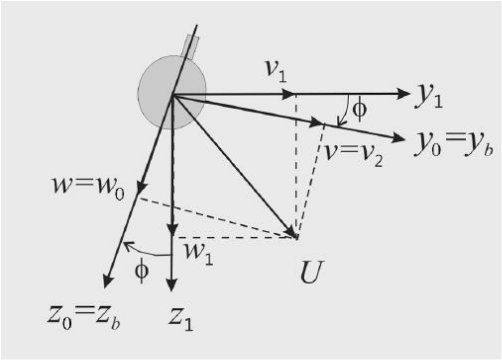


[1] Thor Fossen, "Handbook of Marine Craft Hydrodynamics and Motion Control", Wiley, 2011

Linear velocity transformation

Expanding $$\mathbf{R}_b^n(\phi,\theta,\psi) = \mathbf{R}_{z,\psi}\mathbf{R}_{y,\theta}\mathbf{R}_{x,\phi}$$

$$\mathbf{R}_b^n = \begin{bmatrix} \cos\psi\cos\theta & -\sin\psi\cos\phi + \cos\psi\sin\theta\sin\phi & \sin\phi\sin\psi + \cos\psi\cos\phi\sin\theta \\ \sin\psi\cos\theta & \cos\psi\cos\phi + \sin\psi\sin\theta\sin\phi & -\cos\psi\sin\phi + \sin\psi\cos\phi\sin\theta \\ -\sin\theta & \cos\theta\sin\phi & \cos\theta\cos\phi \end{bmatrix}$$

with

$$\mathbf{p}^n = \eta_1 = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Vehicle position in the **{n}** frame

$$\mathbf{v}^b = \nu_1 = \begin{bmatrix} u \\ v \\ b \end{bmatrix}$$

Vehicle linear velocity in **{b}** frame

the body fixed velocity vector $\mathbf{v}^b = \nu_1$ can be expressed in **{n}**

$$\dot{\mathbf{p}}^n = \mathbf{R}_b^n \mathbf{v}^b \quad \text{or} \quad \boxed{\dot{\eta}_1 = \mathbf{R}_b^n \nu_1}$$

Angular velocity transformation

- Recalling

$$\dot{\eta}_2 = [\dot{\phi}, \dot{\theta}, \dot{\psi}]^T$$

Euler angle rate of change

$$\eta_2 = \begin{bmatrix} \phi \\ \theta \\ \psi \end{bmatrix}$$

Euler angles (roll, pitch and yaw)

$$\mathbf{w}^b = \nu_2 = \begin{bmatrix} p \\ q \\ r \end{bmatrix}$$

Vehicle angular velocity in **{b}** frame

- Clearly $\dot{\eta}_2 \neq \nu_2$ since inertial angular rates are defined in terms of components about the global axes

- Euler rate and body fixed angular velocities can be related by a transformation:

$$\dot{\eta}_2 = \mathbf{T}_\omega(\eta_2)\nu_2$$

- Angular velocity can be related to the instantaneous rate of change of angular orientation only after considerations of the intermediate transformations used

- The transformation matrix can be given by:

$$\nu_2 = \begin{bmatrix} \dot{\phi} \\ 0 \\ 0 \end{bmatrix} + \mathbf{R}_{x,\phi}^T \begin{bmatrix} 0 \\ \dot{\theta} \\ 0 \end{bmatrix} + \mathbf{R}_{x,\phi}^T \mathbf{R}_{y,\theta}^T \begin{bmatrix} 0 \\ 0 \\ \dot{\psi} \end{bmatrix} = \mathbf{T}_\omega^{-1}(\eta_2)\dot{\eta}_2$$

6DOF Kinematics

Expanding

$$\mathbf{T}_{\omega}^{-1}(\eta_2) = \begin{bmatrix} 1 & \sin\phi\tan\theta & \cos\phi\tan\theta \\ 0 & \cos\theta & -\sin\phi \\ 0 & \sin\phi/\cos\theta & \cos\phi/\cos\theta \end{bmatrix} \qquad \mathbf{T}_{\omega}(\eta_2) = \begin{bmatrix} 1 & \sin\phi\tan\theta & \cos\phi\tan\theta \\ 0 & \cos\theta & -\sin\phi \\ 0 & \sin\phi/\cos\theta & \cos\phi/\cos\theta \end{bmatrix}$$

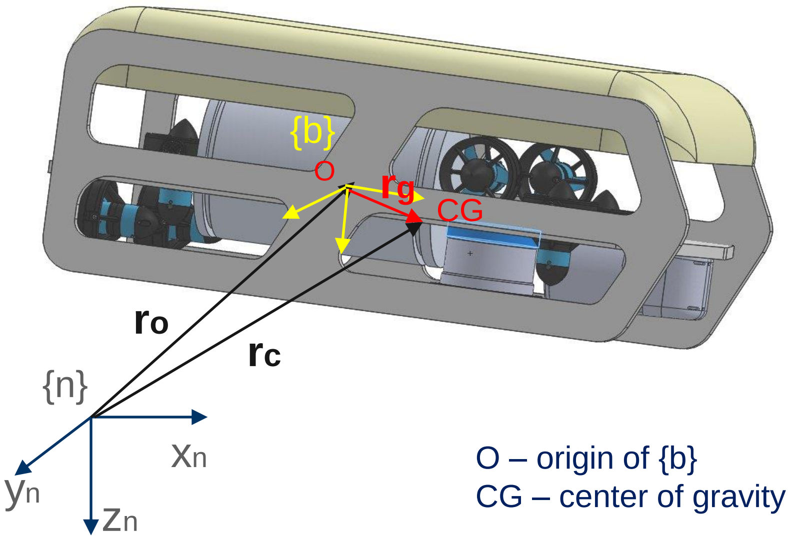
Considering linear and angular velocities the 6 DOF kinematics is

$$\dot{\eta} = \mathbf{J}(\eta)\nu$$

$$\begin{bmatrix} \dot{\eta}_1 \\ \dot{\eta}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{R}_b^n(\eta_2) & \mathbf{0}_{3\times 3} \\ \mathbf{0}_{3\times 3} & \mathbf{T}_{\omega}(\eta_2) \end{bmatrix} \begin{bmatrix} \nu_1 \\ \nu_2 \end{bmatrix}$$

Rigid Body Dynamics

- Vehicle is rigid
- Frame NED is inertial
- Vehicle mass constant



O – origin of {b}
CG – center of gravity

Rigid Body Dynamics

- Differentiation of a vector $\vec{\mathbf{r}}$ in a rotating frame [1,2]

Time derivative in local rotating frame {b}

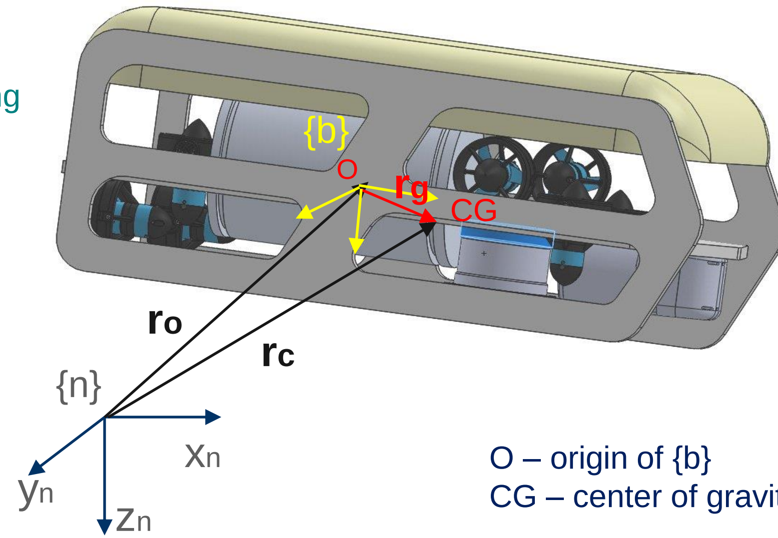
Angular vel. of the rotating frame {b} about {n}

Time derivative in an earth fixed frame {n}

$$\frac{d}{dt}\vec{\mathbf{r}} = \dot{\vec{\mathbf{r}}} + \omega \times \vec{\mathbf{r}}$$

Rate of growth

Rate of transport



O – origin of {b}
CG – center of gravity

- Differentiation of CG vector (rc)

$$\frac{d}{dt}\mathbf{r_c} = \dot{\mathbf{r}}_\mathbf{o} + \omega \times \mathbf{r_g} = \mathbf{v} + \omega \times \mathbf{r_g}$$

- Acceleration

Radial acceleration component

Coriolis acceleration

$$\frac{d^2}{dt}\mathbf{r_c} = \dot{\mathbf{v}} + \dot{\omega} \times \mathbf{r_g} + \omega \times \omega \times \mathbf{r_g} + \omega \times \mathbf{v}$$

Tangential acceleration component

Centripetal acceleration

$\mathbf{v}$ Velocity of CO in {b} given by u,v,w

ω Angular velocity fo {b} relative to {n} fiven by p,q,r

[1] Goldstein, Classical Mechanics, 3ed, Addison Wesley, 2000
[2] A. Healey, “Dynamics and Control of Mobile Robotic Vehicles”, NPS, 1995

Translation motion

- Newton's second law

$$\mathbf{F} = m\frac{d^2\mathbf{r}_c}{dt^2}$$

The translation motion equations relative to the origin of {b}

$$\boxed{\mathbf{F}_o = m(\dot{\mathbf{v}} + \dot{\omega} \times \mathbf{r}_g + \omega \times \omega \times \mathbf{r}_g + \omega \times \mathbf{v})}$$

$\mathbf{F_o}$ Force applied to the body decomposed in the {b} frame

Inertia Matrix

- Sum of applied moments about the rigid body center of mass is equal to the rate of change of angular momentum of the body about **CG**

- It is convenient to consider the point **O** as origin and evaluate the mass moments of inertia about **{b}** (typically **{b}** is chosen along the vehicle principal axes of symmetry)

- Defining the inertia matrix $\mathbf{I}_O$ about **O**

$$\mathbf{I}_o := \begin{bmatrix} I_x & -I_{xy} & -I_{xz} \\ -I_{yx} & I_y & -I_{yz} \\ -I_{zx} & -I_{zy} & I_z \end{bmatrix} \qquad \mathbf{I}_o = \mathbf{I}_o^T > 0$$

ρ_m Body mass density

V Body volume

Moments of inertia about the {b} axes

$$I_x = \int_V (y^2 + z^2)\ \rho_m dV$$
$$I_y = \int_V (x^2 + z^2)\ \rho_m dV$$
$$I_z = \int_V (x^2 + y^2)\ \rho_m dV$$

Products of inertia

$$I_{xy} = \int_V xy\ \rho_m dV = \int_V yx\ \rho_m dV = I_{yx}$$
$$I_{xz} = \int_V xz\ \rho_m dV = \int_V zx\ \rho_m dV = I_{zx}$$
$$I_{yz} = \int_V yz\ \rho_m dV = \int_V zy\ \rho_m dV = I_{zy}$$

- $\mathbf{I}_O$ in vectorial form

$$\mathbf{I}_O \omega = \int_V \mathbf{r} \times (\omega \times \mathbf{r}) \rho_m dV$$

Inertia Matrix

- The inertia matrix $\mathbf{I_c}$ about CG can be transformed to the {b} frame with the parallel axis theorem (Steiner theorem)

$$\mathbf{I}_O = \mathbf{I}_c - m\mathbf{S}^2(\mathbf{r_g}) = \mathbf{I}_c - m(\mathbf{r_g}\mathbf{r_g^T} - \mathbf{r_g}\mathbf{r_g^T}\mathbf{I_{3\times 3}})$$

Example: cylinder radius R and height H

$$dV = dxdydz = (rdrd\theta)dz$$

Polar coordinates

$$x = r\cos\theta, \quad y = r\sin\theta$$

Moment of inertia about CG

$$\mathbf{I}_{zCG} = \int_V (x^2+y^2)dxdydz = \int_{-\frac{H}{2}}^{\frac{H}{2}}\int_0^{2\pi}\int_0^R r^3 drd\theta = \frac{\pi}{2}R^4 H$$

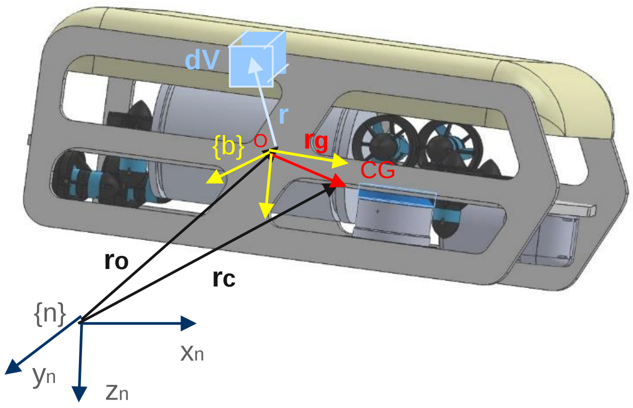
Moment of inertia about O

$$\mathbf{I}_{zO} = \mathbf{I}_{CG} + m(x_g^2 + y_g^2)$$

Rotational motion

- Angular moment about O

$$\mathbf{H}_o = \mathbf{I}_O \omega = \int_V \mathbf{r} \times \mathbf{v} \rho_m dV$$



- Differentiating with respect to time

$$\dot{\mathbf{H}}_O = \underbrace{\int_V \mathbf{r} \times \dot{\mathbf{v}} \rho_m dV}_{\mathbf{M}_O} + \int_V \dot{\mathbf{r}} \times \mathbf{v} \rho_m dV$$

$$\mathbf{M}_O = \int_V \mathbf{r} \times \dot{\mathbf{v}} \rho_m dV$$ Moment about O

- From the figure $\mathbf{v} = \dot{\mathbf{r}}_\mathbf{O} + \dot{\mathbf{r}} \quad \Longrightarrow \quad \dot{\mathbf{r}}_O = \mathbf{v} - \mathbf{v}_O$

- In the time derivative of angular momentum ($\mathbf{v} \times \mathbf{v} = \mathbf{0}$) comes

$$\dot{\mathbf{H}}_O = \mathbf{M}_O - \mathbf{v}_O \times \int_V \mathbf{v} \rho_m dV$$

$$\dot{\mathbf{H}}_O = \mathbf{M}_O - \mathbf{v}_O \times \int_V (\mathbf{v}_O + \dot{\mathbf{r}}) \rho_m dV = \mathbf{M}_O - \mathbf{v}_O \times \int_V \dot{\mathbf{r}} \rho_m dV \quad (1)$$

Rotational motion

- Noting that the distance from O to CG (rg) is given by

$$\mathbf{r}_g = \frac{1}{m}\int_V \mathbf{r}\rho_m dV$$

- Differentiating to time

$$\dot{\mathbf{r}}_g = \frac{1}{m}\int_V \dot{\mathbf{r}}\rho_m dV$$

Since

$$\dot{\mathbf{r}}_g = \omega \times \mathbf{r}_g$$

$\mathbf{r_g}$ does not have rate of change

One can write

$$\int_V \dot{\mathbf{r}}\rho_m dV = m(\omega \times \mathbf{r_g}) \quad (2)$$

- Substituting (2) in (1)

$$\dot{\mathbf{H}}_O = \mathbf{M}_O - m\mathbf{v}_O \times (\omega \times \mathbf{r_g}) \quad (3)$$

- The absolute angular momentum can be written as

$$\mathbf{H}_o = \int_V \mathbf{r} \times \mathbf{v}\rho_m dV = \int_V \mathbf{r} \times \mathbf{v}_O\rho_m dV + \int_V \mathbf{r} \times (\omega \times \mathbf{r})\rho_m$$

$$= \left(\int_V \mathbf{r}\rho_m dV\right) \times \mathbf{v}_O = m\mathbf{r}_g \times \mathbf{v}_O$$

- then

$$\mathbf{H}_O = \mathbf{I}_O\omega + m\mathbf{r}_g \times \mathbf{v}_O$$

Rotational motion

- Differentiating **Ho**, and noting as seen earlier $\frac{d}{dt}\vec{\mathbf{r}} = \dot{\vec{\mathbf{r}}} + \omega \times \vec{\mathbf{r}}$

$$\frac{d}{dt}\mathbf{H}_O = \mathbf{I}_O\dot{\omega} + \omega \times (\mathbf{I}_O\omega) + m(\omega \times \mathbf{r}_g) \times \mathbf{v}_O + m\mathbf{r}_g \times (\dot{\mathbf{v}}_O + \omega \times \mathbf{v}_O) \qquad (4)$$

- Using $(\omega \times \mathbf{r}_g) \times \mathbf{v_O} = -\mathbf{v_O} \times (\omega \times \mathbf{r_g})$ and eliminating the angular moment derivative from (3) and (4)

$$\mathbf{M}_O = \mathbf{I}_O\dot{\omega} + \omega \times (\mathbf{I}_O\omega) + m\mathbf{r}_g) \times (\dot{\mathbf{v}}_O + \omega \times \mathbf{v}_O)$$

Rotational motion

- Angular momentum of the body

$$\mathbf{H}_o = \mathbf{I}_O \omega$$

- From Newton's second law

$$\mathbf{M}_O = \frac{d\mathbf{H}_O}{dt} + \underbrace{\mathbf{r}_g \times (m\frac{d^2\mathbf{r}_c}{dt^2})}$$

Moment of the inertial reaction of sum of external forces acting on the vehicle

- Since **Ho** is given in **{b}** components, and the body is rotating with velocity ω, the rate of growth and rate of transport of the time derivative of the angular momentum must be computed

$$\frac{d\mathbf{H}_O}{dt} = \mathbf{I}_O \dot{\omega} + \omega \times (\mathbf{H}_O)$$

- The rotational equation of motion comes

$$\mathbf{M}_O = \mathbf{I}_O \dot{\omega} + \omega \times (\mathbf{I}_O \omega) + m[\mathbf{r}_g \times \dot{\mathbf{v}}_O + \mathbf{r}_g \times \omega \times \mathbf{v}_O)]$$

$$\boxed{\mathbf{M}_O = \mathbf{I}_O \dot{\omega} + \omega \times (\mathbf{I}_O \omega) + m\mathbf{r}_g \times (\dot{\mathbf{v}}_O + \omega \times \mathbf{v}_O)}$$

6DOF Rigid-body Equations of motion

- The 3 translation and 3 rotation equations of equations of motion

$$\mathbf{F}_o = m(\dot{\mathbf{v}} + \dot{\omega} \times \mathbf{r}_g + \omega \times \omega \times \mathbf{r}_g + \omega \times \mathbf{v})$$

$$\mathbf{M}_O = \mathbf{I}_O\dot{\omega} + \omega \times (\mathbf{I}_O\omega) + m\mathbf{r}_g \times (\dot{\mathbf{v}}_O + \omega \times \mathbf{v}_O)$$

$\mathbf{v} = \nu_1 = [u, v, w]^T$ Velocity in {b} frame

$\omega = \nu_2 = [p, q, r]^T$ Angular velocity of {b} relative to {n}frame

$\mathbf{F}_O = \tau_1 = [X, Y, Z]^T$ Applied total force decomposed in {b}

$\mathbf{M}_O = \tau_2 = [K, M, N]^T$ Applied total moment decomposed in {b}

$\mathbf{r}_g = [x_g, y_g, z_g]^T$ Position of CG in {b}

[1] “Nomenclature for Treating the Motion of a Submerged Body Through a Fluid” The Society of Naval Architects and Marine Engineers, Technical and Research Bulletin No.1--5, April 1950, pp.1--15

6DOF Rigid-body Equations of motion

- In component form

$$
\begin{aligned}
m\left[\dot{u} - vr + wq - x_g(q^2 + r^2) + y_g(pq - \dot{r}) + z_g(pr + \dot{q})\right] &= X \\
m\left[\dot{v} - wp + ur - y_g(r^2 + p^2) + z_g(qr - \dot{p}) + x_g(qp + \dot{r})\right] &= Y \\
m\left[\dot{w} - uq + vp - z_g(p^2 + q^2) + x_g(rp - \dot{q}) + y_g(rq + \dot{p})\right] &= Z
\end{aligned}
$$

Translational motion

$$
\begin{aligned}
I_x\dot{p} + (I_z - I_y)qr - (\dot{r} + pq)I_{xz} + (r^2 - q^2)I_{yz} + (pr - \dot{q})I_{xy} & \\
+m\left[y_g(\dot{w} - uq + vp) - z_g(\dot{v} - wp + ur)\right] &= K \\
I_y\dot{q} + (I_x - I_z)rp - (\dot{p} + qr)I_{xy} + (p^2 - r^2)I_{zx} + (qp - \dot{r})I_{yz} & \\
+m\left[z_g(\dot{u} - vr + wq) - x_g(\dot{w} - uq + vp)\right] &= M \\
I_z\dot{r} + (I_y - I_x)pq - (\dot{q} + rp)I_{yz} + (q^2 - p^2)I_{xy} + (rq - \dot{p})I_{zx} & \\
+m\left[x_g(\dot{v} - wp + ur) - y_g(\dot{u} - vr + wq)\right] &= N
\end{aligned}
$$

Rotational motion

[1] "Nomenclature for Treating the Motion of a Submerged Body Through a Fluid" The Society of Naval Architects and Marine Engineers, Technical and Research Bulletin No.1--5, April 1950, pp.1--15

6DOF Rigid-body Equations of motion

- Vectorial representation [1]

$$\mathrm{M}_{RB}\dot{\boldsymbol{\nu}} + \mathrm{C}_{RB}(\boldsymbol{\nu})\boldsymbol{\nu} = \boldsymbol{\tau}_{RB}$$

Generalized velocity vector expressed in {b}

$$\boldsymbol{\nu} = [u, v, w, p, q, r]^\top$$

Generalized vector of external forces and moments

$$\boldsymbol{\tau}_{RB} = [X, Y, Z, K, M, N]^\top$$

[1] Thor Fossen, “Guidance and Control of Ocean Vehicles”, Wiley, 1994

Inertia matrix M_{RB}

- Constant and positive definite

$$\mathbf{M}_{RB} = \mathbf{M}_{RB}^T > 0 \qquad \dot{\mathbf{M}}_{RB} = 0$$

with

$$\mathbf{M}_{RB} = \begin{bmatrix} m\mathbf{I}_{3\times3} & -m\mathbf{S}(\mathbf{r}_g) \\ m\mathbf{S}(\mathbf{r}_g) & \mathbf{I}_O \end{bmatrix}$$

$$= \begin{bmatrix} m & 0 & 0 & 0 & mz_g & -my_g \\ 0 & m & 0 & -mz_g & 0 & mx_g \\ 0 & 0 & m & my_g & -mx_g & 0 \\ 0 & -mz_g & my_g & I_x & -I_{xy} & -I_{xz} \\ mz_g & 0 & -mx_g & -I_{yx} & I_y & -I_{yz} \\ -my_g & mx_g & 0 & -I_{zx} & -I_{zy} & I_z \end{bmatrix}$$

[1] Thor Fossen, “Guidance and Control of Ocean Vehicles”, Wiley, 1994

Coriolis-Centripetal Matrix C_{RB}

- The $\mathbf{C}_{RB}$ matrix can be represented in a skew-symmetrical form [1]

$$\mathrm{C}_{RB}(\boldsymbol{\nu}) = -\mathrm{C}_{RB}^{T}(\boldsymbol{\nu}) \qquad \forall \boldsymbol{\nu} \in \mathbb{R}^6$$

With a M inertia matrix as

$$\mathrm{M} = \mathrm{M}^T = \begin{bmatrix} \mathrm{M}_{11} & \mathrm{M}_{12} \\ \mathrm{M}_{21} & \mathrm{M}_{22} \end{bmatrix} > 0$$

C can be parameterized such that is skew-symmetrical [1]

$$\mathrm{C}(\boldsymbol{\nu}) = \begin{bmatrix} 0_{3\times3} & -\mathrm{S}(\mathrm{M}_{11}\boldsymbol{\nu}_1 + \mathrm{M}_{12}\boldsymbol{\nu}_2) \\ -\mathrm{S}(\mathrm{M}_{11}\boldsymbol{\nu}_1 + \mathrm{M}_{12}\boldsymbol{\nu}_2) & -\mathrm{S}(\mathrm{M}_{21}\boldsymbol{\nu}_1 + \mathrm{M}_{22}\boldsymbol{\nu}_2) \end{bmatrix}$$

$$\boldsymbol{\nu}_1 = [u, v, w]^T$$

$$\boldsymbol{\nu}_2 = [p, q, r]^T$$

[1] S. Sagatun and T. Fossen, “Lagrangian formulation od underwater vehicles dynamics”, Proc. IEEE Int. Conf. Systems, Man and Cybernetics, Charlosttesville, USA, 1991

Coriolis-Centripetal Matrix

- For the rigid body coriolis and centripetal matrix $\mathbf{C}_{RB}(\boldsymbol{\nu}) = -\mathbf{C}_{RB}^{T}(\boldsymbol{\nu}) \quad \forall \boldsymbol{\nu} \in \mathbb{R}^6$

$$\mathbf{C}_{RB}(\boldsymbol{\nu}) = \begin{bmatrix} \mathbf{0}_{3\times 3} & -m\mathbf{S}(\boldsymbol{\nu}_1) - m\mathbf{S}(\mathbf{S}(\boldsymbol{\nu}_2)\mathbf{r}_g) \\ -m\mathbf{S}(\boldsymbol{\nu}_1) - m\mathbf{S}(\mathbf{S}(\boldsymbol{\nu}_2)\mathbf{r}_g) & m\mathbf{S}(\mathbf{S}(\boldsymbol{\nu}_1)\mathbf{r}_g) - \mathbf{S}(\mathbf{I}_o\boldsymbol{\nu}_2) \end{bmatrix}$$

in component form

$$\boldsymbol{C}_{RB}(\boldsymbol{\nu}) = \begin{bmatrix} 0 & 0 & 0 & m(y_g q + z_g r) & -m(x_g q - w) & -m(x_g r + v) \\ 0 & 0 & 0 & -m(y_g p + w) & m(z_g r + x_g p) & -m(y_g r - u) \\ 0 & 0 & 0 & -m(z_g p - v) & -m(z_g q + u) & m(x_g p + y_g q) \\ -m(y_g q + z_g r) & m(y_g p + w) & m(z_g p - v) & 0 & -I_{yz} q - I_{xz} p + I_z r & I_{yz} r + I_{xy} p - I_y q \\ m(x_g q - w) & -m(z_g r + x_g p) & m(z_g q + u) & I_{yz} q + I_{xz} p - I_z r & 0 & -I_{xz} r - I_{xy} q + I_x p \\ m(x_g r + v) & m(y_g r - u) & -m(x_g p + y_g q) & -I_{yz} r - I_{xy} p + I_y q & I_{xz} r + I_{xy} q - I_x p & 0 \end{bmatrix}$$

Skew-symmetry of CRB is useful since it implies $\boldsymbol{\nu}^{\top}\boldsymbol{C}_{RB}(\boldsymbol{\nu})\boldsymbol{\nu} \equiv 0$. This can be used in nonlinear control design using Lyapunov functions with energy based designs.

[1] Thor Fossen, “Handbook of Marine Craft Hydrodynamics and Motion Control”, Wiley, 2011

Coriolis-Centripetal Matrix

- A linear velocity independent parametrization can be obtained [1]

$$C_{RB}(\boldsymbol{\nu}) = \begin{bmatrix} m\boldsymbol{S}(\boldsymbol{\nu}_2) & -m\boldsymbol{S}(\boldsymbol{\nu}_2)\boldsymbol{S}(\boldsymbol{r}_G) \\ m\boldsymbol{S}(\boldsymbol{r}_G)\boldsymbol{S}(\boldsymbol{\nu}_2) & -\boldsymbol{S}(\boldsymbol{I}_0\,\boldsymbol{\nu}_2) \end{bmatrix}$$

This is useful since independence of linear velocity ($\nu_1 = [u\ v\ w]^T$) can be exploited in the case of the vehicle being under irrotational ocean currents. In this case [2]:

$$\boldsymbol{M}_{RB}\dot{\boldsymbol{\nu}} + \boldsymbol{C}_{RB}(\boldsymbol{\nu})\boldsymbol{\nu} \equiv \boldsymbol{M}_{RB}\dot{\boldsymbol{\nu}}_r + \boldsymbol{C}_{RB}(\boldsymbol{\nu}_r)\boldsymbol{\nu}_r$$

$$\boldsymbol{\nu}_r = \boldsymbol{\nu} - \boldsymbol{\nu}_c$$ Relative velocity

$$\boldsymbol{\nu}_c := [u_c, v_c, w_c, 0, 0, 0]^\top$$

[1] T. Fossen, and O. Fjellstad ,"Nonlinear modelling of Marine Vehicles in 6 Degrees of Freedom", Journal of Mathematical Modelling of Systems, vol1, n1, 1995

[2] Thor Fossen, "Handbook of Marine Craft Hydrodynamics and Motion Control", Wiley, 2011

Example

- Calculation with MSS toolbox [1] function `m2c.m`

```
% rigid-body system inertia matrix
MRB = [1000*eye(3)  zeros(3,3)
        zeros(3,3)  10000*eye(3)];

% rigid-body Coriolis and centripetal matrix
nu  = [10 1 1 1 2 3]'
CRB = m2c(MRB,nu)
```

$$\boldsymbol{C}_{RB} = \begin{bmatrix} 0 & 0 & 0 & 0 & 1000 & -1000 \\ 0 & 0 & 0 & -1000 & 0 & 10000 \\ 0 & 0 & 0 & 1000 & -10000 & 0 \\ 0 & 1000 & -1000 & 0 & 30000 & -20000 \\ -1000 & 0 & 10000 & -30000 & 0 & 10000 \\ 1000 & -10000 & 0 & 20000 & -10000 & 0 \end{bmatrix}$$

In T. Fossen, “Handbook of Marine Craft Hydrodynamics and Motion Control”, Wiley, 2011

6DOF Equations of Motion

- Recalling, the vehicle kinematics

$$\dot{\eta} = \mathbf{J}(\eta)\nu$$

- And as seen earlier the 6DOF rigid body equations of motion are in vectorial form

$$\mathbf{M}_{RB}\dot{\boldsymbol{\nu}} + \mathbf{C}_{RB}(\boldsymbol{\nu})\boldsymbol{\nu} = \boldsymbol{\tau}_{RB}$$

Where the external generalized forces and moments vector τ_{RB} accounts for all the external forces/moments applied to the body.

This vector includes hydrostatic effects (buoyancy and weight) , propulsion, all the hydrodynamic forces and environment disturbances (such as wind, waves etc)

$$\tau_{RB} = \tau_{rest} + \tau_{hydro} + \tau_{wind} + \tau_{wave} + \tau$$

- This global model can be shown [1] to be in the form

$$\begin{aligned} &\dot{\eta} = \mathbf{J}(\eta)\nu \\ &\mathbf{M}\dot{\nu} + \mathbf{C}(\nu)\nu + \mathbf{D}(\nu)\nu + \mathbf{g}(\eta) = \tau + \tau_{\mathbf{wind}} + \tau_{\mathbf{wave}} \end{aligned} \qquad (*)$$

[1] Thor Fossen, “Guidance and Control of Ocean Vehicles”, Wiley, 1994

Hydrostatics and hydrodynamics

Buoyancy: A force on a submerged body acting at its volumetric center equal to the weight of displaced fluid, $B = \rho g V$

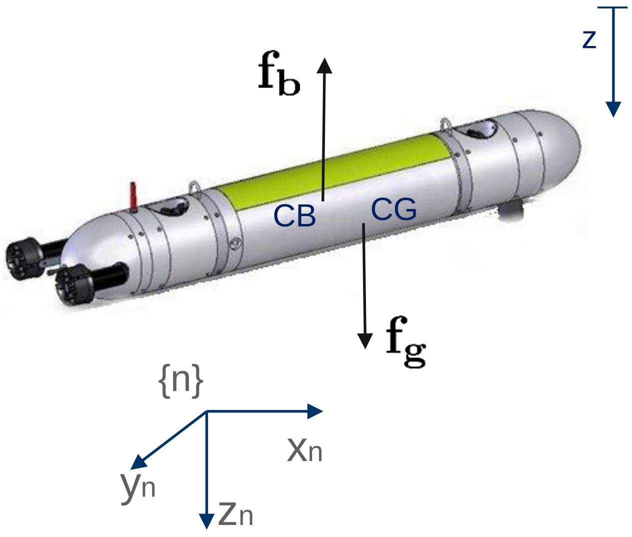
Drag: Force caused by viscous and fluid separation effects
$F = 0.5 \rho C_d A_f u_r |u_r|$

Lift: Force caused by fluid circulation over surfaces
$F = 0.5 \rho C_L A_f u_r |u_r|; \; C_L = \frac{dC_L}{d\alpha} \alpha$

Inertia: Force caused by inertial reaction between body and fluid resulting from relative acceleration and representing the effects of stored energy in the surrounding fluid

Hydrostatics

- Gravitational and buoyancy forces are called **restoring forces** in hydrostatics
- Restoring forces depend on the volume of displaced water, location of center of buoyancy, gravity and mass
- Two classes of vehicles
  - Underwater vehicles
  - Surface vessels



Hydrostatics - Underwater vehicles

- Weight and buoyancy force are given by:

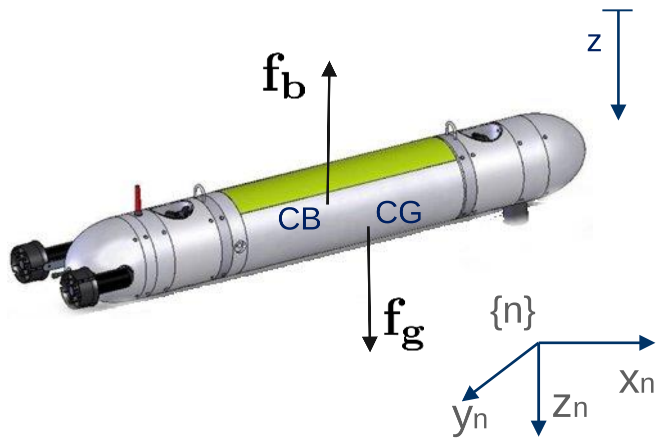
$$W = mg$$

$$B = \rho g \nabla$$

∇ Displaced volume

ρ Water density

g Acceleration of gravity



The gravity and buoyancy forces are applied in the z axis (vertical) of the inertial frame {n} (z positive downwards, so buoyancy negative and gravity positive), so in this frame:

$$f_g^n = \begin{bmatrix} 0 \\ 0 \\ W \end{bmatrix} \qquad f_b^n = -\begin{bmatrix} 0 \\ 0 \\ B \end{bmatrix}$$

The forces can be expressed in {b} using the rotation matrix $\mathbf{R}_n^b(\eta_2) = \mathbf{R}_b^n(\eta_2)^{-1}$

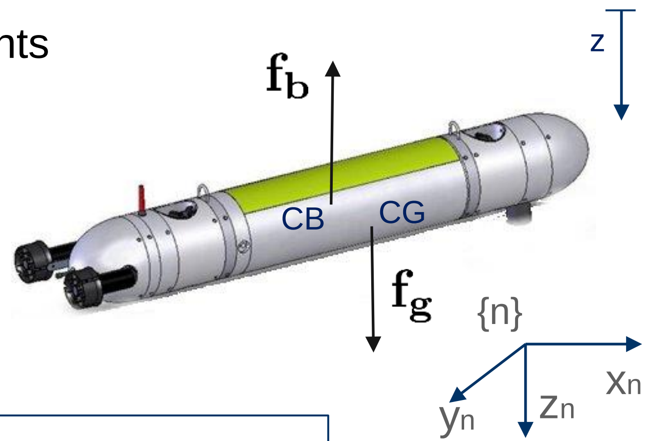
$$\mathbf{f}_g^b = \mathbf{R}_n^b(\eta_2)\mathbf{f}_g^n$$ Weight force

$$\mathbf{f}_b^b = \mathbf{R}_n^b(\eta_2)\mathbf{f}_b^n$$ Buoyancy force

Hydrostatics - Underwater vehicles

- In the body fixed frame the restoring forces and moments are

$$\tau_{\mathbf{rest}} = \begin{bmatrix} \mathbf{f}_g^b + \mathbf{f}_b^b \\ \mathbf{r}_g^b \times \mathbf{f}_g^b + \mathbf{r}_b^b \times \mathbf{f}_b^b \end{bmatrix}$$



- Considering the general vectorial expression [1]

$$\dot{\eta} = \mathbf{J}(\eta)\nu$$

$$\mathbf{M}\dot{\nu} + \mathbf{C}(\nu)\nu + \mathbf{D}(\nu)\nu + \mathbf{g}(\eta) = \tau + \tau_{\mathbf{wind}} + \tau_{\mathbf{wave}}$$

- The restoring force/moment term $\mathbf{g}(\eta)$ comes: $\mathbf{g}(\eta) = -\tau_{\mathbf{rest}}$

$$g(\eta) = \begin{bmatrix} (W-B)\ \sin(\theta) \\ -\ (W-B)\ \cos(\theta)\sin(\phi) \\ -\ (W-B)\ \cos(\theta)\cos(\phi) \\ -\ (y_g W - y_b B)\ \cos(\theta)\cos(\phi) + (z_g W - z_b B)\ \cos(\theta)\sin(\phi) \\ (z_g W - z_b B)\ \sin(\theta) + (x_g W - x_b B)\ \cos(\theta)\cos(\phi) \\ -\ (x_g W - x_b B)\ \cos(\theta)\sin(\phi) - (y_g W - y_b B)\ \sin(\theta) \end{bmatrix}$$

[1] Thor Fossen, “Guidance and Control of Ocean Vehicles”, Wiley, 1994

Example

- Using MSS toolbox `gvect` to compute $\mathbf{g}(\eta)$

```
r_g = [0, 0, 0]        % location of CG with respect to CO
r_b = [0, 0, -10]      % location of CB with respect to CO

m   = 1000             % mass
g   = 9.81             % acceleration of gravity
W   = m*g;             % weight
B   = W;               % buoyancy

% pitch and roll angles
theta = 10*(180/pi); phi = 30*(pi/180)

% g-vector
g = gvect(W,B,theta,phi,r_g,r_b)
```

$$\boldsymbol{g} = 10^4 \cdot [0, 0, 0, 1.8324, 9.0997, 0]^\top$$

In T. Fossen, “Handbook of Marine Craft Hydrodynamics and Motion Control”, Wiley, 2011

Hydrostatics – Surface vehicles

- Static stability due to restoring forces is denominated as **metacentric stability** in hydrostatics for surface vessels

- A vessel is metacentric stable if can resist disturbances in heave, roll and pitch from its equilibrium point

- Restoring forces in this case depend on metacentric height, location of CB, CG and shape and size of water plane

A_{wp} Water plane area

$\overline{GM}_T$ Transverse metacentric height

$\overline{GM}_L$ Longitudinal metacentric height

Metacenter – point where am imaginary vertical line trough CB intersects another imaginary vertical line trough a new CB when the vessel is displaced or tilted in the water

Hydrostatics – Surface vehicles

- Transverse metacentric stability

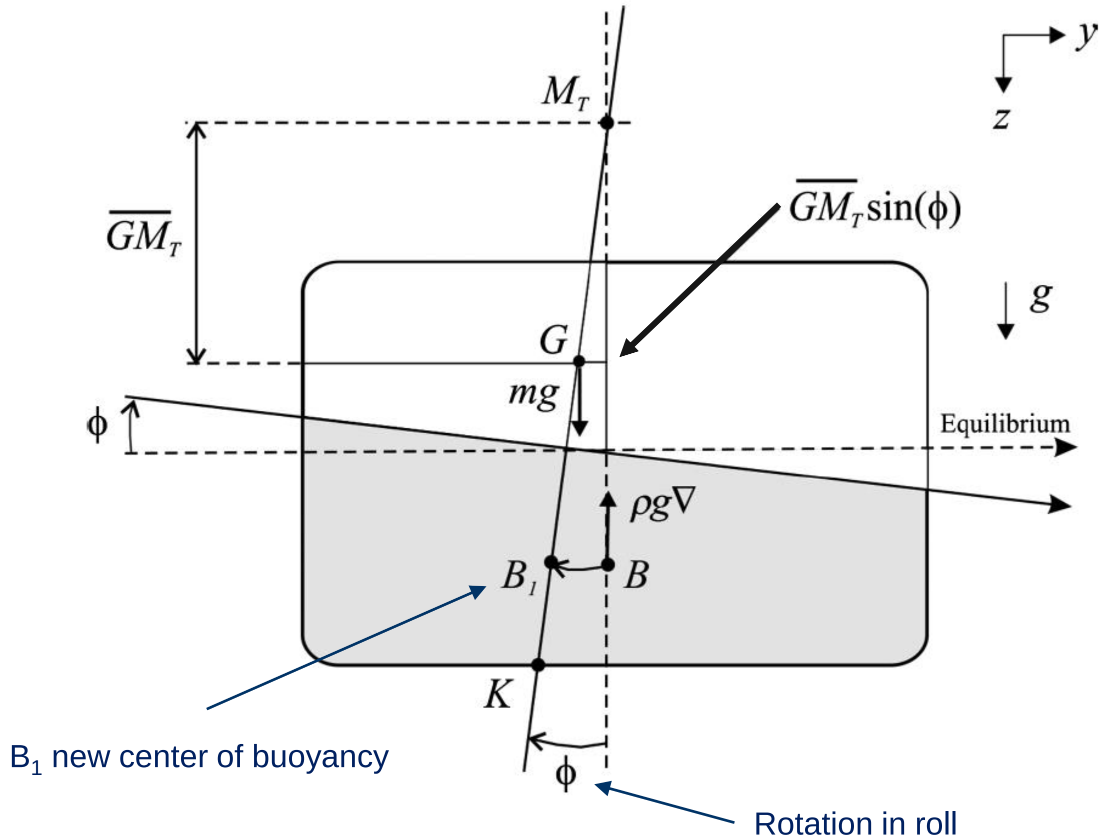


Figure from T. Fossen, “Handbook of Marine Craft Hydrodynamics and Motion Control”, Wiley, 2011

Hydrostatics – Surface vehicles

- In unperturbed equilibrium

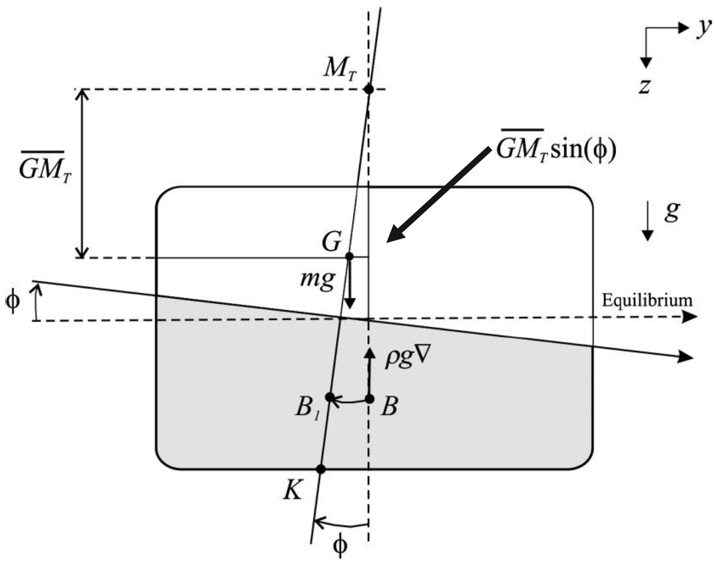
$$mg = \rho g \nabla$$

- Force in heave Z is given by

$$Z = mg - \rho g \left[\nabla + \delta\nabla(z)\right]$$
$$= -\rho g \delta\nabla(z)$$

with

$$\delta\nabla(z) = \int_0^z A_{wp}(\zeta)\mathrm{d}\zeta$$



z	displacement in heave
$z = 0$	equilibrium position for the nominal displacement volume ∇
$\delta\nabla(z)$	displacement volume due to perturbation in heave
$A_{wp}(\zeta)$	water plane area as function of heave position

Figure from T. Fossen, “Handbook of Marine Craft Hydrodynamics and Motion Control”, Wiley, 2011

Hydrostatics – Surface vehicles

- For conventional ships it can be assumed small variations of water plane area for small perturbations in z

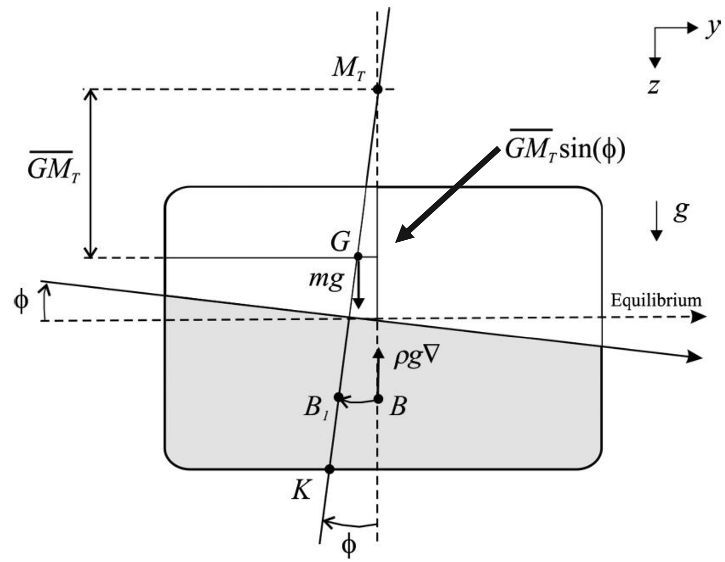
$$A_{wp}(\zeta) \approx A_{wp}(0)$$

then the restoring force is linear in z

$$Z \approx -\rho g A_{wp}(0) \; z$$

- The restoring force $\delta \mathbf{f}^b_{rest}$ expressed in **{b}** is then

$$\begin{aligned} \delta \mathbf{f}^b_{rest} &= \mathbf{R}^b_n \delta \mathbf{f}^n_{rest} \\ &= \mathbf{R}^b_n \begin{bmatrix} 0 \\ 0 \\ -\rho g \int_0^z A_{wp}(\zeta) d\zeta \end{bmatrix} \\ &= -\rho g \begin{bmatrix} -\sin(\theta) \\ \cos(\theta)\sin(\phi) \\ \cos(\theta)\cos(\phi) \end{bmatrix} \int_0^z A_{wp}(\zeta) d\zeta \end{aligned}$$



z displacement in heave

$z = 0$ equilibrium position for the nominal displacement volume ∇

$\delta\nabla(z)$ displacement volume due to perturbation in heave

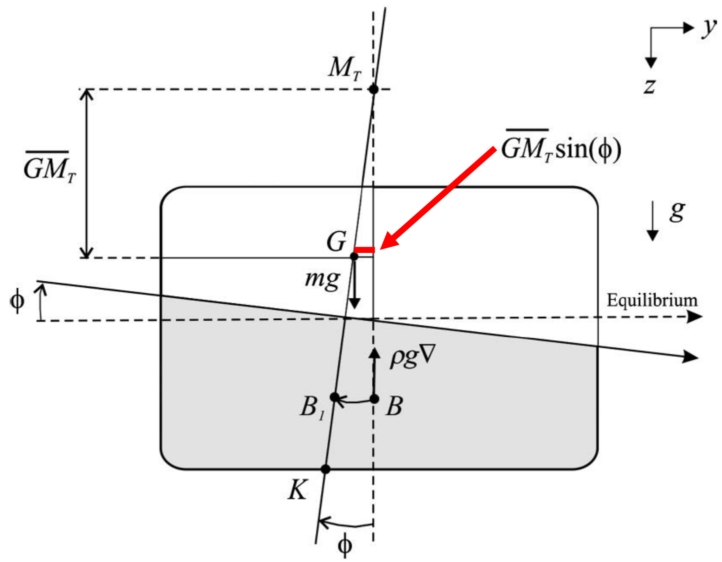
$A_{wp}(\zeta)$ water plane area as function of heave position

Figure from T. Fossen, “Handbook of Marine Craft Hydrodynamics and Motion Control”, Wiley, 2011

Hydrostatics – Surface vehicles

- The restoring moments in roll and pith are related with the moment arm $\overline{GM}_L \sin(\theta)$ and $\overline{GM}_T \sin(\phi)$ (see figure in red)

- The moment is created by a force pair in the z direction with W=B



Then

$$\mathbf{r}^b_{rest} = \begin{bmatrix} -\overline{GM}_L \sin(\theta) \\ \overline{GM}_T \sin(\phi) \\ 0 \end{bmatrix}$$

$$\delta \mathbf{f}^b_{rest} = \mathbf{R}^b_n \begin{bmatrix} 0 \\ 0 \\ -\rho g \nabla \end{bmatrix} = -\rho g \nabla \begin{bmatrix} -\sin(\theta) \\ \cos(\theta)\sin(\phi) \\ \cos(\theta)\cos(\phi) \end{bmatrix}$$

- The restoring moment is thus (neglecting the moment due $\mathbf{r}^b_{rest} \times \delta \mathbf{f}^b_{rest} = 0$ to perturbation in z)

$$\mathbf{m}^b_{rest} = \mathbf{r}^b_{rest} \times \mathbf{f}^b_{rest}$$

$$= -\rho g \nabla \begin{bmatrix} \overline{GM}_T \sin(\phi)\cos(\theta)\cos(\phi) \\ \overline{GM}_L \sin(\theta)\cos(\theta)\cos(\phi) \\ (-\overline{GM}_L \cos(\theta) + \overline{GM}_T)\sin(\phi)\sin(\theta) \end{bmatrix}$$

Hydrostatics –Surface vehicles

- Then the restoring forces and moments are

$$\tau_{rest} = \begin{bmatrix} \delta \mathbf{f}^b_{rest} \\ \mathbf{m}^b_{rest} \end{bmatrix}$$

so the term $\mathbf{g}(\eta)$ in expression (*) is

$$\mathbf{g}(\eta) = -\begin{bmatrix} \delta \mathbf{f}^b_{rest} \\ \mathbf{m}^b_{rest} \end{bmatrix}$$

In component form

$$g(\eta) = \begin{bmatrix} -\rho g \int_0^z A_{wp}(\zeta) \mathrm{d}\zeta \ \sin(\theta) \\ \rho g \int_0^z A_{wp}(\zeta) \mathrm{d}\zeta \ \cos(\theta)\sin(\phi) \\ \rho g \int_0^z A_{wp}(\zeta) \mathrm{d}\zeta \ \cos(\theta)\cos(\phi) \\ \rho g \nabla \overline{GM}_T \sin(\phi)\cos(\theta)\cos(\phi) \\ \rho g \nabla \overline{GM}_L \sin(\theta)\cos(\theta)\cos(\phi) \\ \rho g \nabla (-\overline{GM}_L \cos\theta + \overline{GM}_T) \sin(\phi)\sin(\theta) \end{bmatrix}$$

Hydrodynamics

- The hydrodynamic forces acting on a body moving in a fluid require the knowledge of the velocity and pressure of the fluid at each location
- Given a fluid velocity vector $\boldsymbol{v}(\boldsymbol{x}, t) = [v_1(\boldsymbol{x}, t),\ v_2(\boldsymbol{x}, t),\ v_3(\boldsymbol{x}, t)]^\top$
- Considering an incompressible fluid the continuity equation

$$\operatorname{div}(\boldsymbol{v}) = \nabla \cdot \boldsymbol{v} = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z} = 0$$

- And the forces can be obtained from solving the conservation of moment in the flow given by the **Navier-Stokes equations** subject to the continuity equation and boundary conditions

$$\rho \left(\frac{\partial \boldsymbol{v}}{\partial t} + \boldsymbol{v} \cdot \nabla \boldsymbol{v} \right) = \rho \boldsymbol{F} - \nabla p + \mu \nabla^2 \boldsymbol{v}$$

- This forms a system of non-linear partial differential equations that can only be solved numerically. In general the hydrodynamics modelling is obtained with approximations and simplifying assumptions.

See [1] for a detailed introduction to marine hydrodynamics or other classic fluid mechanics references

[1] J. Newman, “Marine Hydrodynamics”, MIT Press, 1977

Hydrodynamics

- The *maneuvering theory* assumes that hydrodynamic forces and moments can be linearly superposed and are independent of frequency (without wave excitation)

See [1] and [2] for a ***seakeeping theory*** treatment of vehicle motion in the presence of waves

- Hydrodynamic forces can be grouped in
  - **Radiation induced forces** – body forced to oscillate with wave excitation frequency and no incident waves
  - **Diffraction forces** – body is retrained to oscillate with incident waves

- Hydrodynamic forces τ_{hydro} group a set of effects, namely added mass effects, hydrodynamic damping (usually designated as drag due to multiple causes) and lifting forces.

[1] T. Perez, " Ship Motion Control", Springer 2005
[2] T. Fossen, "Hanbook of Marine craft hydrodynamics and motion control", Wiley,. 2011

Hydrodynamics

- Hydrodynamic forces and moments τ_{hydro} are constituted [1] by
  - Added mass effects
  - Potential damping
  - Skin friction damping
  - Wave drift damping
  - Vortex shedding damping

$$\tau_{hydro} = \underbrace{-\mathbf{M}_A\dot{\boldsymbol{\nu}} - \mathbf{C}_A(\boldsymbol{\nu})\boldsymbol{\nu}}_{\text{Added mass}} \underbrace{- \mathbf{D}_P(\boldsymbol{\nu})\boldsymbol{\nu}}_{\text{Potential damping}} \underbrace{- \mathbf{D}_S(\boldsymbol{\nu})\boldsymbol{\nu}}_{\text{Skin friction}} \underbrace{- \mathbf{D}_W(\boldsymbol{\nu})\boldsymbol{\nu}}_{\text{Wave drift damping}} \underbrace{- \mathbf{D}_M(\boldsymbol{\nu})\boldsymbol{\nu}}_{\text{Vortex shedding}}$$

total hydrodynamic damping matrix $\mathbf{D}(\boldsymbol{\nu})$

$$\mathbf{D}(\boldsymbol{\nu}) := \mathbf{D}_P(\boldsymbol{\nu}) + \mathbf{D}_S(\boldsymbol{\nu}) + \mathbf{D}_W(\boldsymbol{\nu}) + \mathbf{D}_M(\boldsymbol{\nu})$$

[1] Thor Fossen, “Guidance and Control of Ocean Vehicles”, Wiley, 1994

Hydrodynamics

- Recalling the rigid body 6DOF model

$$\mathbf{M}_{RB}\dot{\boldsymbol{\nu}} + \mathbf{C}_{RB}(\boldsymbol{\nu})\boldsymbol{\nu} = \boldsymbol{\tau}_{RB}$$

$$\tau_{RB} = \tau_{rest} + \tau_{hydro} + \tau_{wind} + \tau_{wave} + \tau$$

- Resulting model

$$\mathbf{M}\dot{\nu} + \mathbf{C}(\nu)\nu + \mathbf{D}(\nu)\nu + \mathbf{g}(\eta) = \tau + \tau_{\mathbf{wind}} + \tau_{\mathbf{wave}}$$

with

$$\mathbf{M} = \mathbf{M}_{RB} + \mathbf{M}_A$$

$$\mathbf{C}(\boldsymbol{\nu}) = \mathbf{C}_{RB}(\boldsymbol{\nu}) + \mathbf{C}_A(\boldsymbol{\nu})$$

Hydrodynamics – currents

- General model

$$\mathbf{M}\dot{\nu} + \mathbf{C}(\nu)\nu + \mathbf{D}(\nu)\nu + \mathbf{g}(\eta) = \tau + \tau_{\mathbf{wind}} + \tau_{\mathbf{wave}}$$

with irrotational ocean currents, the relative velocity is defined

$$\boldsymbol{\nu}_r = \boldsymbol{\nu} - \boldsymbol{\nu}_c$$
$$\boldsymbol{\nu}_c = [u_c, v_c, w_c, 0, 0, 0]^\top$$

then the model becomes

$$\mathbf{M_{RB}}\dot{\nu} + \mathbf{C_{RB}}(\nu)\nu + \mathbf{M_A}\dot{\nu}_\mathbf{r} + \mathbf{C_A}(\nu_\mathbf{r})\nu_\mathbf{r} + \mathbf{D}(\nu_\mathbf{r})\nu_\mathbf{r} + \mathbf{g}(\eta) = \tau + \tau_{\mathbf{wind}} + \tau_{\mathbf{wave}}$$

- If currents are constant and irrotational in {n}

$$\boxed{\mathbf{M}\dot{\nu}_\mathbf{r} + \mathbf{C}(\nu_\mathbf{r})\nu_\mathbf{r} + \mathbf{D}(\nu_\mathbf{r})\nu_\mathbf{r} + \mathbf{g}(\eta) = \tau + \tau_{\mathbf{wind}} + \tau_{\mathbf{wave}}}$$

Added Mass

- Inertia effects caused by the motion of fluid surrounding the vehicle

- Usually considered as an extra mass of water to be added to the original mass (with effect in the acceleration components).

- The assumption of a finite extra mass of water is not entirely correct since the vehicle motion will force all the fluid to oscillate with different amplitudes

- Added mass depend only on the body geometry [1] under the assumptions of
  - Ideal irrotational fluid neglecting viscous forces
  - Body rigid and constant volume
  - Fluid unbounded and infinite

[1] J. Newman, “Marine Hydrodynamics”, MIT Press, 1977

Added Mass

- The motion of the vehicle through the fluid causes the existence of a kinetic energy in the fluid given by [1]

$$T_A = \frac{1}{2}\boldsymbol{\nu}^T \mathrm{M}_A \boldsymbol{\nu}$$

with M_A a 6x6 added mass inertia matrix

$$\mathrm{M}_A = \begin{bmatrix} \mathrm{A}_{11} & \mathrm{A}_{12} \\ \mathrm{A}_{21} & \mathrm{A}_{22} \end{bmatrix} = - \begin{bmatrix} X_{\dot{u}} & X_{\dot{v}} & X_{\dot{w}} & X_{\dot{p}} & X_{\dot{q}} & X_{\dot{r}} \\ Y_{\dot{u}} & Y_{\dot{v}} & Y_{\dot{w}} & Y_{\dot{p}} & Y_{\dot{q}} & Y_{\dot{r}} \\ Z_{\dot{u}} & Z_{\dot{v}} & Z_{\dot{w}} & Z_{\dot{p}} & Z_{\dot{q}} & Z_{\dot{r}} \\ K_{\dot{u}} & K_{\dot{v}} & K_{\dot{w}} & K_{\dot{p}} & K_{\dot{q}} & K_{\dot{r}} \\ M_{\dot{u}} & M_{\dot{v}} & M_{\dot{w}} & M_{\dot{p}} & M_{\dot{q}} & M_{\dot{r}} \\ N_{\dot{u}} & N_{\dot{v}} & N_{\dot{w}} & N_{\dot{p}} & N_{\dot{q}} & N_{\dot{r}} \end{bmatrix}$$

where the hydrodynamic coefficients represent the added mass force/moment due to acceleration in the corresponding directions

$$Y = -\underbrace{Y_{\dot{u}}}\dot{u}$$

Added mass force **Y** (along *yb* axis) due to $\dot{u}$ (acceleration in *xb* axis)

Added mass coefficient

$$Y_{\dot{u}} := \frac{\partial Y}{\partial \dot{u}}$$

Added mass derivative (SNAME notation [1])

[1] "Nomenclature for Treating the Motion of a Submerged Body Through a Fluid" The Society of Naval Architects and Marine Engineers, Technical and Research Bulletin No.1--5, April 1950, pp.1--15

Inertial force for a cylinder model

Stationary Fluid:

$$X_f = -C_a \rho(\pi c^2)\dot{u}_r$$

Accelerating Fluid -
No Body Acceleration:

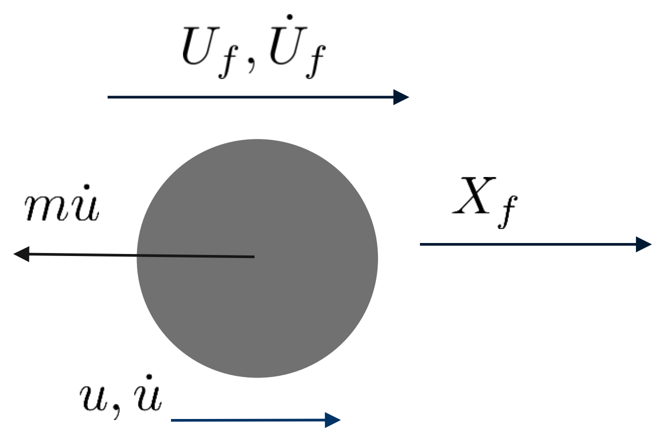
$$\rho \pi c^2 (1 + C_a)\dot{U}_f$$

Fluid Acceleration
plus Body Acceleration:

$$X_f = [\rho(\pi c^2) + C_a \rho(\pi c^2)]\dot{U}_f - C_a \rho(\pi c^2)\dot{u}$$

Relative velocity:

$$u_r = u - U_f$$



Added Mass

- For underwater vehicles

$$\mathbf{M}_A > 0$$

- For bodies at rest the added mass matrix is positive definite

In many practical cases the off diagonal elements of $\mathbf{M_A}$ can be neglected when comparing with the diagonal ones

$$\mathbf{M}_A = \mathbf{M}_A^T > 0$$

- Added mass forces and moments can be derived from the Kirchoff's equations [1]

$$\begin{aligned} \frac{d}{dt}\left(\frac{\partial T}{\partial \boldsymbol{\nu}_1}\right) + \mathbf{S}(\boldsymbol{\nu}_2)\frac{\partial T}{\partial \boldsymbol{\nu}_1} &= \boldsymbol{\tau}_1 \\ \frac{d}{dt}\left(\frac{\partial T}{\partial \boldsymbol{\nu}_2}\right) + \mathbf{S}(\boldsymbol{\nu}_2)\frac{\partial T}{\partial \boldsymbol{\nu}_2} + \mathbf{S}(\boldsymbol{\nu}_1)\frac{\partial T}{\partial \boldsymbol{\nu}_1} &= \boldsymbol{\tau}_2 \end{aligned}$$

[1] Thor Fossen, “Guidance and Control of Ocean Vehicles”, Wiley, 1994

Added Mass

Added mass terms in component form [1]

$$
\begin{aligned}
X_A &= X_{\dot u}\dot u + X_{\dot w}(\dot w + uq) + X_{\dot q}\dot q + Z_{\dot w}wq + Z_{\dot q}q^2 && \text{longitudinal elements (line 1)}\\
&\quad + X_{\dot v}\dot v + X_{\dot p}\dot p + X_{\dot r}\dot r - Y_{\dot v}vr - Y_{\dot p}rp - Y_{\dot r}r^2 && \text{lateral elements (line 2)}\\
&\quad - X_{\dot v}ur - Y_{\dot w}wr && \text{mixed u or w (line 3)}\\
&\quad + Y_{\dot w}vq + Z_{\dot p}pq - (Y_{\dot q} - Z_{\dot r})qr && \text{mixed terms (usually neglected), (line 4)}\\
Y_A &= X_{\dot v}\dot u + Y_{\dot w}\dot w + Y_{\dot q}\dot q\\
&\quad + Y_{\dot v}\dot v + Y_{\dot p}\dot p + Y_{\dot r}\dot r + X_{\dot v}vr - Y_{\dot w}vp + X_{\dot r}r^2 + (X_{\dot p} - Z_{\dot r})rp - Z_{\dot p}p^2\\
&\quad - X_{\dot w}(up - wr) + X_{\dot u}ur - Z_{\dot w}wp\\
&\quad - Z_{\dot q}pq + X_{\dot q}qr\\
Z_A &= X_{\dot w}(\dot u - wq) + Z_{\dot w}\dot w + Z_{\dot q}\dot q - X_{\dot u}uq - X_{\dot q}q^2\\
&\quad + Y_{\dot w}\dot v + Z_{\dot p}\dot p + Z_{\dot r}\dot r + Y_{\dot v}vp + Y_{\dot r}rp + Y_{\dot p}p^2\\
&\quad + X_{\dot v}up + Y_{\dot w}wp\\
&\quad - X_{\dot v}vq - (X_{\dot p} - Y_{\dot q})pq - X_{\dot r}qr\\
K_A &= X_{\dot p}\dot u + Z_{\dot p}\dot w + K_{\dot q}\dot q - X_{\dot v}wu + X_{\dot r}uq - Y_{\dot w}w^2 - (Y_{\dot q} - Z_{\dot r})wq + M_{\dot r}q^2\\
&\quad + Y_{\dot p}\dot v + K_{\dot p}\dot p + K_{\dot r}\dot r + Y_{\dot w}v^2 - (Y_{\dot q} - Z_{\dot r})vr + Z_{\dot p}vp - M_{\dot r}r^2 - K_{\dot q}rp\\
&\quad + X_{\dot w}uv - (Y_{\dot v} - Z_{\dot w})vw - (Y_{\dot r} + Z_{\dot q})wr - Y_{\dot p}wp - X_{\dot q}ur\\
&\quad + (Y_{\dot r} + Z_{\dot q})vq + K_{\dot r}pq - (M_{\dot q} - N_{\dot r})qr\\
M_A &= X_{\dot q}(\dot u + wq) + Z_{\dot q}(\dot w - uq) + M_{\dot q}\dot q - X_{\dot w}(u^2 - w^2) - (Z_{\dot w} - X_{\dot u})wu\\
&\quad + Y_{\dot q}\dot v + K_{\dot q}\dot p + M_{\dot r}\dot r + Y_{\dot p}vr - Y_{\dot r}vp - K_{\dot r}(p^2 - r^2) + (K_{\dot p} - N_{\dot r})rp\\
&\quad - Y_{\dot w}uv + X_{\dot v}vw - (X_{\dot r} + Z_{\dot p})(up - wr) + (X_{\dot p} - Z_{\dot r})(wp + ur)\\
&\quad - M_{\dot r}pq + K_{\dot q}qr\\
N_A &= X_{\dot r}\dot u + Z_{\dot r}\dot w + M_{\dot r}\dot q + X_{\dot v}u^2 + Y_{\dot w}wu - (X_{\dot p} - Y_{\dot q})uq - Z_{\dot p}wq - K_{\dot q}q^2\\
&\quad + Y_{\dot r}\dot v + K_{\dot r}\dot p + N_{\dot r}\dot r - X_{\dot v}v^2 - X_{\dot r}vr - (X_{\dot p} - Y_{\dot q})vp + M_{\dot r}rp + K_{\dot q}p^2\\
&\quad - (X_{\dot u} - Y_{\dot v})uv - X_{\dot w}vw + (X_{\dot q} + Y_{\dot p})up + Y_{\dot r}ur + Z_{\dot q}wp\\
&\quad - (X_{\dot q} + Y_{\dot p})vq - (K_{\dot p} - M_{\dot q})pq - K_{\dot r}qr
\end{aligned}
$$

Including
added mass inertia
and
coriolis and centripetal
components

[1] F. Imlay, "The complete expressions for added mass of a rigid body moving in an ideal fluid", Tech report DTMB 1528, David Taylor Model Basin, USA, 1961

Added Mass

- Added mass coriolis and centripetal matrix can be written[1] as

$$\mathbf{C}_A(\boldsymbol{\nu}) = -\mathbf{C}_A^T(\boldsymbol{\nu})$$

$$\mathbf{C}_A(\boldsymbol{\nu}) = \begin{bmatrix} \mathbf{0}_{3\times 3} & -\mathbf{S}(\mathbf{A}_{11}\boldsymbol{\nu}_1 + \mathbf{A}_{12}\boldsymbol{\nu}_2) \\ -\mathbf{S}(\mathbf{A}_{11}\boldsymbol{\nu}_1 + \mathbf{A}_{12}\boldsymbol{\nu}_2) & -\mathbf{S}(\mathbf{A}_{21}\boldsymbol{\nu}_1 + \mathbf{A}_{22}\boldsymbol{\nu}_2 \end{bmatrix} \qquad \mathbf{M}_A = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix}$$

in component form

$$\mathbf{C}_A(\boldsymbol{\nu}) = \begin{bmatrix} 0 & 0 & 0 & 0 & -a_3 & a_2 \\ 0 & 0 & 0 & a_3 & 0 & -a_1 \\ 0 & 0 & 0 & -a_2 & a_1 & 0 \\ 0 & -a_3 & a_2 & 0 & -b_3 & b_2 \\ a_3 & 0 & -a_1 & b_3 & 0 & -b_1 \\ -a_2 & a_1 & 0 & -b_2 & b_1 & 0 \end{bmatrix}$$

$$\begin{aligned} a_1 &= X_{\dot{u}}u + X_{\dot{v}}v + X_{\dot{w}}w + X_{\dot{p}}p + X_{\dot{q}}q + X_{\dot{r}}r \\ a_2 &= X_{\dot{v}}u + Y_{\dot{v}}v + Y_{\dot{w}}w + Y_{\dot{p}}p + Y_{\dot{q}}q + Y_{\dot{r}}r \\ a_3 &= X_{\dot{w}}u + Y_{\dot{w}}v + Z_{\dot{w}}w + Z_{\dot{p}}p + Z_{\dot{q}}q + Z_{\dot{r}}r \\ b_1 &= X_{\dot{p}}u + Y_{\dot{p}}v + Z_{\dot{p}}w + K_{\dot{p}}p + K_{\dot{q}}q + K_{\dot{r}}r \\ b_2 &= X_{\dot{q}}u + Y_{\dot{q}}v + Z_{\dot{q}}w + K_{\dot{q}}p + M_{\dot{q}}q + M_{\dot{r}}r \\ b_3 &= X_{\dot{r}}u + Y_{\dot{r}}v + Z_{\dot{r}}w + K_{\dot{r}}p + M_{\dot{r}}q + N_{\dot{r}}r \end{aligned}$$

[1] T. Fossen, “Nonlinear modeling and control of underwater vehicles”, PhD. Thesis, NTNU 1991

- Lyapunov analysis exploiting model properties [1]

Model

$$\dot{\boldsymbol{\eta}} = \boldsymbol{J}_k(\boldsymbol{\eta})\boldsymbol{\nu}$$

$$\boldsymbol{M}\dot{\boldsymbol{\nu}} + \boldsymbol{C}(\boldsymbol{\nu})\boldsymbol{\nu} + \boldsymbol{D}(\boldsymbol{\nu})\boldsymbol{\nu} + \boldsymbol{g}(\boldsymbol{\eta}) = \boldsymbol{\tau}$$

Lyapunov function candidate (based on kinetic and potential energy)

$$V = \frac{1}{2}\boldsymbol{\nu}^\top \boldsymbol{M}\boldsymbol{\nu} + \frac{1}{2}\boldsymbol{\eta}^\top \boldsymbol{K}_p\boldsymbol{\eta}$$

constant gain matrix

$$\boldsymbol{K}_p = \boldsymbol{K}_p^\top > 0$$

Since $\boldsymbol{M} = \boldsymbol{M}^\top > 0$, $\dot{\boldsymbol{M}} = \boldsymbol{0}$ then

$$\begin{aligned} \dot{V} &= \boldsymbol{\nu}^\top \boldsymbol{M}\dot{\boldsymbol{\nu}} + \boldsymbol{\eta}^\top \boldsymbol{K}_p\dot{\boldsymbol{\eta}} \\ &= \boldsymbol{\nu}^\top \boldsymbol{M}\dot{\boldsymbol{\nu}} + \boldsymbol{\eta}^\top \boldsymbol{K}_p\boldsymbol{J}_k(\boldsymbol{\eta})\boldsymbol{\nu} \\ &= \boldsymbol{\nu}^\top [\boldsymbol{M}\dot{\boldsymbol{\nu}} + \boldsymbol{J}_k^\top(\boldsymbol{\eta})\boldsymbol{K}_p\boldsymbol{\eta}] \end{aligned}$$

[1] ex.7.8 from Thor Fossen, “Handbook of Marine Craft Hydrodynamics and Motion Control”, Wiley

Substituting the model velocity eq. gives

$$\begin{aligned}\dot{V} &= \boldsymbol{\nu}^{\top}[\boldsymbol{\tau} - \boldsymbol{C}(\boldsymbol{\nu})\boldsymbol{\nu} - \boldsymbol{D}(\boldsymbol{\nu})\boldsymbol{\nu} - \boldsymbol{g}(\boldsymbol{\eta}) + \boldsymbol{J}_k^{\top}(\boldsymbol{\eta})\boldsymbol{K}_p\boldsymbol{\eta}] \\ &= \nu^{\mathbf{T}}\tau - \nu^{\mathbf{T}}\mathbf{C}(\nu)\nu - \nu^{\mathbf{T}}\mathbf{D}(\nu)\nu + \nu^{\mathbf{T}}\mathbf{g}(\eta)\mathbf{J}_{\mathbf{k}}^{\mathbf{T}}(\eta)\mathbf{K}_{\mathbf{p}}\eta\end{aligned}$$

Since $\boldsymbol{\nu}^{\top}\boldsymbol{C}(\boldsymbol{\nu})\boldsymbol{\nu} \equiv \mathbf{0}$, $\boldsymbol{\nu}^{\top}\boldsymbol{D}(\boldsymbol{\nu})\boldsymbol{\nu} > 0$

A PD control law can be chosen

$$\boldsymbol{\tau} = \boldsymbol{g}(\boldsymbol{\eta}) - \boldsymbol{K}_d\boldsymbol{\nu} - \boldsymbol{J}_k^{\top}(\boldsymbol{\eta})\boldsymbol{K}_p\boldsymbol{\eta}$$

that renders the derivative of V semi-negative, with $\boldsymbol{K}_d > 0$

$$\begin{aligned}\dot{V} &= -\boldsymbol{\nu}^{\top}[\boldsymbol{K}_d + \boldsymbol{D}(\boldsymbol{\nu})]\boldsymbol{\nu} \\ &\leq 0\end{aligned}$$

Added Mass – Example surface vehicles

- For surface vehicles with xz symmetry, the surge can be decoupled from steering

$$\boldsymbol{\nu}_r = [u_r, v_r, r]^\top$$

- The added mass derivatives for surge, sway and yaw motions , neglecting heave, pith and roll modes, for high speed (U>>0) $\mathrm{M}_A(U) \neq \mathrm{M}_A^T(U)$

$$\begin{bmatrix} X \\ Y \\ N \end{bmatrix} = \begin{bmatrix} X_{\dot{u}} & 0 & 0 \\ 0 & Y_{\dot{v}} & Y_{\dot{r}} \\ 0 & N_{\dot{v}} & N_{\dot{r}} \end{bmatrix} \begin{bmatrix} \dot{u} \\ \dot{v} \\ \dot{r} \end{bmatrix} + \begin{bmatrix} 0 & 0 & -Y_{\dot{v}}v - \frac{1}{2}\underbrace{(Y_{\dot{r}} + N_{\dot{v}})}_{Y_{\dot{r}}} r \\ 0 & 0 & X_{\dot{u}}u \\ Y_{\dot{v}}v + \frac{1}{2}\underbrace{(Y_{\dot{r}} + N_{\dot{v}})}_{Y_{\dot{r}}} r & -X_{\dot{u}}u & 0 \end{bmatrix} \begin{bmatrix} u \\ v \\ r \end{bmatrix}$$

- For low speed $\mathrm{M}_A = \mathrm{M}_A^T \Rightarrow Y_{\dot{r}} = N_{\dot{v}}$

$$\boldsymbol{M}_A = \boldsymbol{M}_A^\top = -\begin{bmatrix} X_{\dot{u}} & 0 & 0 \\ 0 & Y_{\dot{v}} & Y_{\dot{r}} \\ 0 & Y_{\dot{r}} & N_{\dot{r}} \end{bmatrix} \qquad \boldsymbol{C}_A(\boldsymbol{\nu}_r) = -\boldsymbol{C}_A^\top(\boldsymbol{\nu}_r) = \begin{bmatrix} 0 & 0 & Y_{\dot{v}}v_r + Y_{\dot{r}}r \\ 0 & 0 & -X_{\dot{u}}u_r \\ -Y_{\dot{v}}v_r - Y_{\dot{r}}r & X_{\dot{u}}u_r & 0 \end{bmatrix}$$

Munk moment

- Added mass forces and moments for surge, sway and yaw surface vessel at low speed

$$\begin{bmatrix} X \\ Y \\ N \end{bmatrix} = \underbrace{\begin{bmatrix} X_{\dot{u}} & 0 & 0 \\ 0 & Y_{\dot{v}} & Y_{\dot{r}} \\ 0 & N_{\dot{v}} & N_{\dot{r}} \end{bmatrix}}_{\mathbf{M}_A} \begin{bmatrix} \dot{u} \\ \dot{v} \\ \dot{r} \end{bmatrix} + \underbrace{\begin{bmatrix} 0 & 0 & Y_{\dot{v}} v_r + Y_{\dot{r}} r \\ 0 & 0 & -X_{\dot{u}} u_r \\ -Y_{\dot{v}} v_r - Y_{\dot{r}} r & X_{\dot{u}} u_r & 0 \end{bmatrix}}_{\boldsymbol{C}_A(\boldsymbol{v}_r)} \begin{bmatrix} u \\ v \\ r \end{bmatrix}$$

$$\boldsymbol{C}_A(\boldsymbol{v})\boldsymbol{v} = \begin{bmatrix} Y_{\dot{v}} v_r r + Y_{\dot{r}} r^2 \\ -X_{\dot{u}} u_r r \\ \underbrace{(X_{\dot{u}} - Y_{\dot{v}}) u_r v_r}_{\text{Munk moment}} - Y_{\dot{r}} u_r r \end{bmatrix}$$

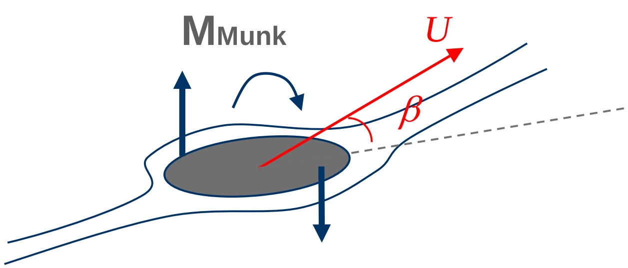
- *Munk moment* is a destabilizing moment arising from the asymmetric location of stagnation points, pressure highest on the front and lowest on the back [1].
- It acts to turn the vehicle perpendicular to the flow

$$M_{Munk} = (X_{\dot{u}} - Y_{\dot{v}}) u_r v_r$$

$$M_{Munk} = -\tfrac{1}{2}(Y_{\dot{v}} - X_{\dot{u}}) \sin 2\beta$$

β sideslip angle

U Speed



[1] M. Triantafyllou, F. Hover, “Maneuvering and control of marine vehicles”, Lecture notes Course 13.49, MIT, 2002

Added Mass – Example underwater vehicles

- For underwater symmetrical vehicles (a mild assumption since in general there can be some asymmetry in the xy and or yz planes) at low speed (such as ROVs) M_A can be considered diagonal

$$\mathrm{M}_A = -\mathrm{diag}\{X_{\dot{u}}, Y_{\dot{v}}, Z_{\dot{w}}, K_{\dot{p}}, M_{\dot{q}}, N_{\dot{r}}\}$$

$$\mathrm{C}_A(\boldsymbol{\nu}) = \begin{bmatrix} 0 & 0 & 0 & 0 & -Z_{\dot{w}}w & Y_{\dot{v}}v \\ 0 & 0 & 0 & Z_{\dot{w}}w & 0 & -X_{\dot{u}}u \\ 0 & 0 & 0 & -Y_{\dot{v}}v & X_{\dot{u}}u & 0 \\ 0 & -Z_{\dot{w}}w & Y_{\dot{v}}v & 0 & -N_{\dot{r}}r & M_{\dot{q}}q \\ Z_{\dot{w}}w & 0 & -X_{\dot{u}}u & N_{\dot{r}}r & 0 & -K_{\dot{p}}p \\ -Y_{\dot{v}}v & X_{\dot{u}}u & 0 & -M_{\dot{q}}q & K_{\dot{p}}p & 0 \end{bmatrix}$$

- The diagonal approximation of M_A is useful since generally the off diagonal elements are small in comparison and their determination is also more difficult.

Damping

$$\tau_{hydro} = -\underbrace{\mathbf{M}_A\dot{\boldsymbol{\nu}} - \mathbf{C}_A(\boldsymbol{\nu})\boldsymbol{\nu}}_{\text{Added mass}} - \overbrace{\underbrace{\mathbf{D}_P(\boldsymbol{\nu})\boldsymbol{\nu}}_{\text{Potential damping}} - \underbrace{\mathbf{D}_S(\boldsymbol{\nu})\boldsymbol{\nu}}_{\text{Skin friction}} - \underbrace{\mathbf{D}_W(\boldsymbol{\nu})\boldsymbol{\nu}}_{\text{Wave drift damping}} - \underbrace{\mathbf{D}_M(\boldsymbol{\nu})\boldsymbol{\nu}}_{\text{Vortex shedding}}}^{\mathbf{D}(\boldsymbol{\nu})}$$

- Potential damping usually negligible comparable with the other terms
- Linear skin friction due the laminar boundary layer, and an additional quadratic term at high speed
- Wave drift damping resistance for surface vehicles advancing in waves (significant in surge for high sea states)
- Vortex shedding caused by vortexes at the edges of the vehicle

Drag

- In a viscous fluid the vortex shedding drag

$$F_{drag}(U) = -\frac{1}{2}\rho\, C_D(\mathrm{Rn})\, A|U|U$$

with the Reynolds number given by

$$\mathrm{Rn} = \frac{UD}{\nu}$$

Ratio of dynamic forces to the friction forces

ρ	Fluid density
R_n	Reynolds number
$C(R_n)$	Drag coefficient
A	Projected cross-section area
U	Vehicle speed
D	Characteristic length of the body
ν	Kinematic viscosity coefficient (1.56 x 10^-6 for salt water at 5ºC with 3.5% salinity)

- In general the above force drag expression can be used to determine the quadratic drag with the appropriate Cd and representative area.
- As a consequence the effective drag force is dependent on the angle of attack/sideslip, Reynolds number and general body shape

[1] M. Triantafyllou, F. Hover, “Maneuvering and control of marine vehicles”, Lecture notes Course 13.49, MIT, 2002

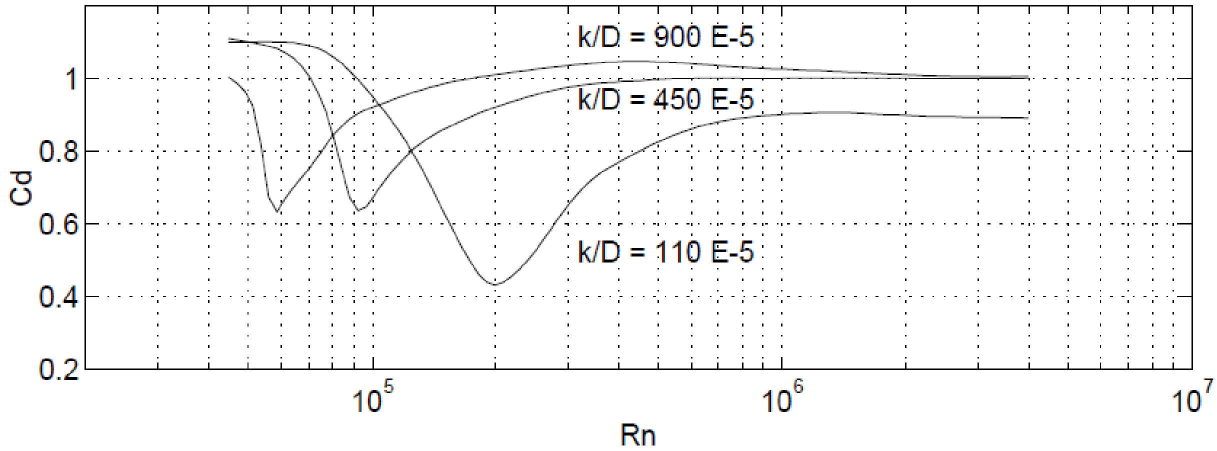
Drag

- Quadratic drag in 6DOF

$$\mathbf{D}_n(\boldsymbol{\nu})\boldsymbol{\nu} = \begin{bmatrix} |\boldsymbol{\nu}|^T \mathbf{D}_{n1}\boldsymbol{\nu} \\ |\boldsymbol{\nu}|^T \mathbf{D}_{n2}\boldsymbol{\nu} \\ |\boldsymbol{\nu}|^T \mathbf{D}_{n3}\boldsymbol{\nu} \\ |\boldsymbol{\nu}|^T \mathbf{D}_{n4}\boldsymbol{\nu} \\ |\boldsymbol{\nu}|^T \mathbf{D}_{n5}\boldsymbol{\nu} \\ |\boldsymbol{\nu}|^T \mathbf{D}_{n6}\boldsymbol{\nu} \end{bmatrix}$$

With Di being in general 6x6 matrices depending on density, Cd and A (both Cd and area will be different for different elements)

Example of Cd curves for rough cylinder under steady flow with surface average roughness k and diameter D, from [1]



[1] O. Faltinsen, “Sea loads on ships and offshore structures”, Cambridge University Press, 1990“Fluid-Dynamic Drag”, 1995

Drag

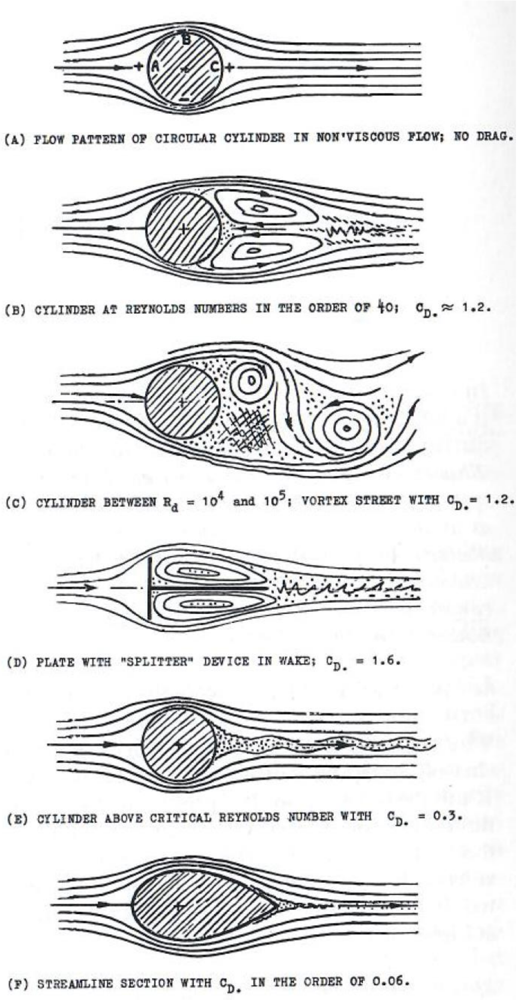
- The type of flow has strong impact in the actual drag force (see figure for 2D drag coef. at different conditions)

- For a first approximation of drag forces calculation there are a number of empirical data tables and formulas, considering different shapes and regimes. See [1] for a comprehensive treatment and data.

  – Example of a particular model (Jorgenson): for streamlined bodies the axial skin friction drag can be obtained by [2]

$$F_x = \frac{\rho}{2} S_{wet} C_f(Rn)[1 + 60(D/L)^3 + 0.0025(L/D)]|u|u$$

L	Length
S_{wet}	Wet area (total)
D	Diameter



(A) FLOW PATTERN OF CIRCULAR CYLINDER IN NON'VISCOUS FLOW; NO DRAG.

(B) CYLINDER AT REYNOLDS NUMBERS IN THE ORDER OF 40; $c_{D.} \approx 1.2$.

(C) CYLINDER BETWEEN $R_d = 10^4$ and 10^5; VORTEX STREET WITH $c_{D.} = 1.2$.

(D) PLATE WITH "SPLITTER" DEVICE IN WAKE; $c_{D.} = 1.6$.

(E) CYLINDER ABOVE CRITICAL REYNOLDS NUMBER WITH $c_{D.} = 0.3$.

(F) STREAMLINE SECTION WITH $c_{D.}$ IN THE ORDER OF 0.06.

Figure from [1]

[1] S. F. Hoerner, "Fluid-Dynamic Drag", 1995

[2] K. Watkinson, N. Smith, D. Henn, "Users manual for TRJUUV: A six degrees of freedom simulation program for unmanned underwater vehicles" ARAP report n. 635, Applied Physics Tenchnology In, 1989 1995

Drag

- In general is difficult so separate the different contributions from the multiple causes to the drag

- It is convenient to write the total hydrodynamic damping as the sum of a linear part with a quadratic one

$$\mathbf{D}(\boldsymbol{\nu}) = \mathbf{D} + \mathbf{D}_n(\boldsymbol{\nu})$$

$\mathbf{D}$ Linear drag

$\mathbf{D}_n(\boldsymbol{\nu})$ Non Linear drag

- In ideal fluid the damping matrix is non-symmetrical and positive

$$\mathbf{D}(\boldsymbol{\nu}) > 0$$

Drag

- Linear drag model for low speed
  - ROVs
  - In hovering or station-keeping conditions
- Quadratic drag at “high” speed
  - Ships, AUVs at cruise speed

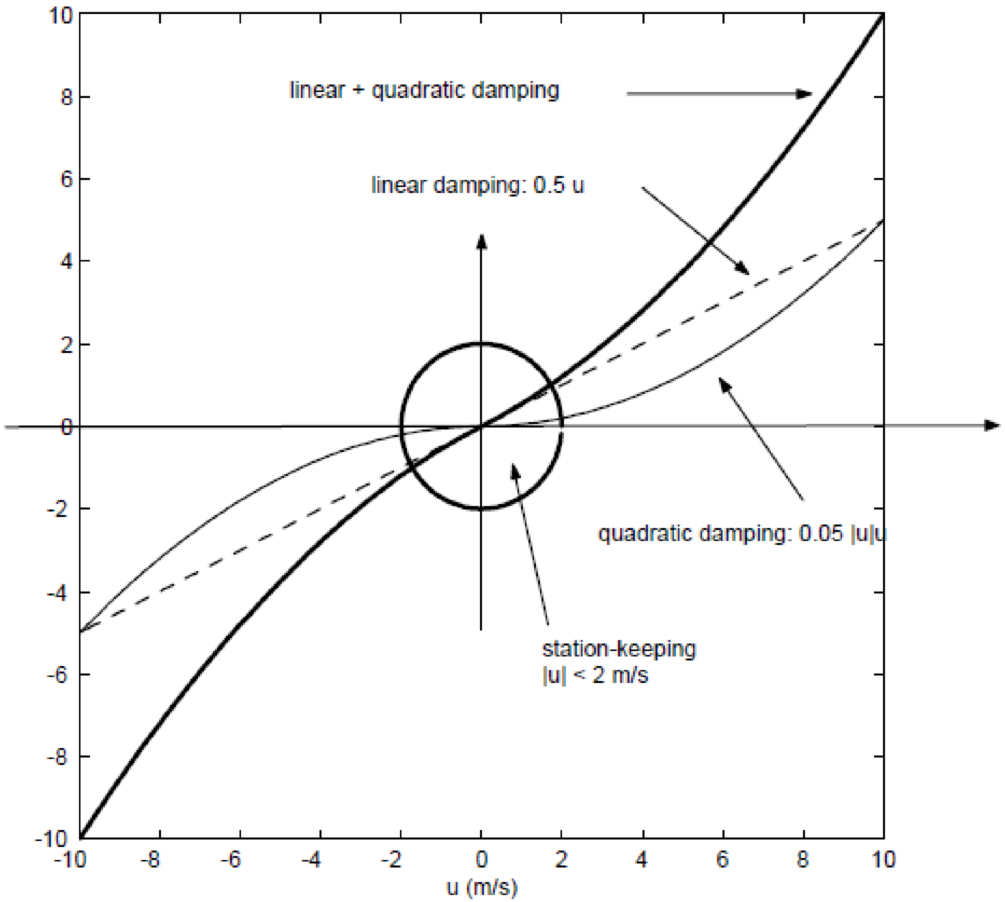


Figure from T. Fossen, “Handbook of Marine Craft Hydrodynamics and Motion Control”, Wiley, 2011

Normalization

- It is common to use the prime system of SNAME [1] to normalize the hydrodynamic coefficients.
- In this system uses velocity U, length L, L/U as time unit and $\frac{\rho}{2}L^3$ or $\frac{\rho}{2}L^2T$ as mass unit
- The non dimensional states are obtained as:

$$\dot{v}' = \frac{L}{U^2}\dot{v} \qquad v' = \frac{1}{U}v \qquad \dot{r}' = \frac{L^2}{U^2}\dot{r} \qquad r' = \frac{L}{U}r$$

- And the constant terms as:

$$m' = \frac{m}{\frac{1}{2}\rho L^3} \qquad I'_{zz} = \frac{I_{zz}}{\frac{1}{2}\rho L^5} \qquad x'_G = \frac{\bar{x}_G}{L} \qquad U' = \frac{U}{U} = 1 \qquad Y'_{\dot{v}} = \frac{Y_{\dot{v}}}{\frac{1}{2}\rho L^3}$$

$$Y'_v = \frac{Y_v}{\frac{1}{2}\rho U L^2} \qquad Y'_{\dot{r}} = \frac{Y_{\dot{r}}}{\frac{1}{2}\rho L^4} \qquad Y'_r = \frac{Y_r}{\frac{1}{2}\rho U L^3} \qquad Y' = \frac{Y}{\frac{1}{2}\rho U^2 L^2} \qquad N'_{\dot{v}} = \frac{N_{\dot{v}}}{\frac{1}{2}\rho L^4}$$

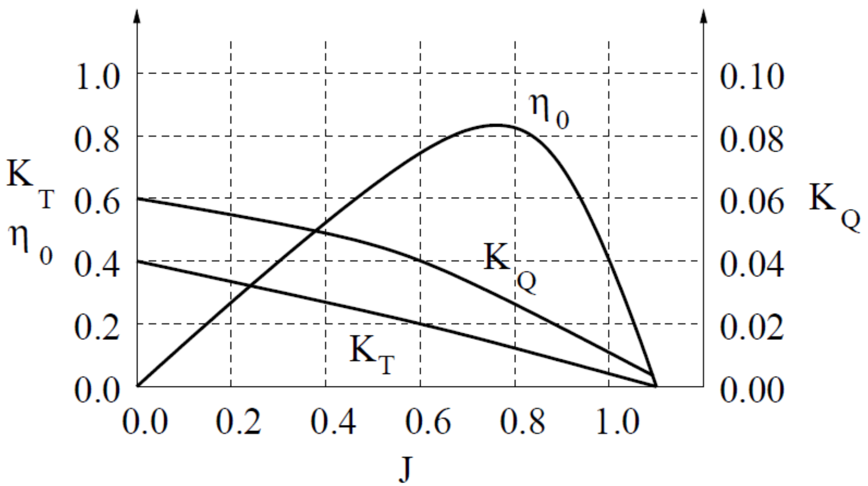
$$N'_v = \frac{N_v}{\frac{1}{2}\rho U L^3} \qquad N'_{\dot{r}} = \frac{N_{\dot{r}}}{\frac{1}{2}\rho L^5} \qquad N'_r = \frac{N_r}{\frac{1}{2}\rho U L^4} \qquad N' = \frac{N}{\frac{1}{2}\rho U^2 L^3}.$$

[1] “Nomenclature for Treating the Motion of a Submerged Body Through a Fluid” The Society of Naval Architects and Marine Engineers, Technical and Research Bulletin No.1--5, April 1950, pp.1--15

Thruster model

- Thruster force depends on propeller speed (n), and advance speed factor (Jo) and propeller diameter (D)

$$F_{thrust} = \underbrace{\rho D^4 K_T(J_o)}_{F_{n|n|}} n|n|$$



- Advance speed factor expresses the relative water flow in the propeller factoring the vehicle velocity

$$J_o = \frac{V_A}{n D}$$

Figure from M. Tryantafillou, “Maneuvering and control of marine vehicles”, MIT, 2002

Thruster model

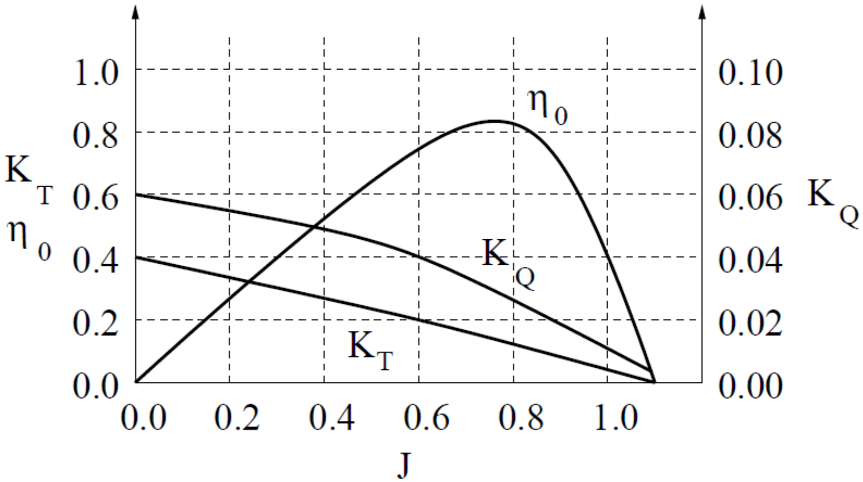
- Hydrodynamic load torque is given by

$$Q = \rho D^5 K_Q(J_o) n |n|$$

- with open propeller efficiency given by

$$\eta_o = \frac{J(U) K_T}{2\pi K_Q}$$

- Multiple thruster models have been presented in the literature. One relevant property is the consideration of non symmetric thrusters.



- Full 6DOF model

$$\dot{\eta} = \mathbf{J}(\eta)\nu$$

$$\mathbf{M}\dot{\nu} + \mathbf{C}(\nu)\nu + \mathbf{D}(\nu)\nu + \mathbf{g}(\eta) = \tau + \tau_{\mathbf{wind}} + \tau_{\mathbf{wave}}$$

Parameter determination

- Vehicle properties determination
  - Classical ship maneuvers

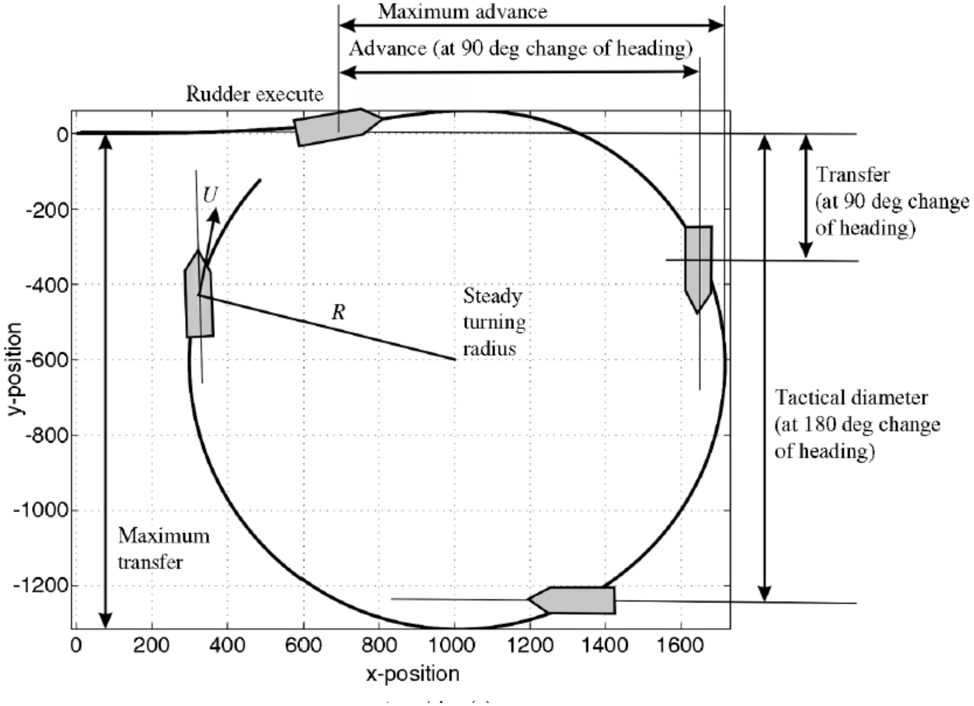
- Vehicle model parameters
  - Empiric calculations / tables
  - CFD - Computer fluid dynamics, panel methods
  - Towing tank tests
  - Open water tests

Standard ship maneuvers

- Standard open loop maneuvers for determining ship maneuverability characteristics
  - Straight line stability
  - Stability on course
  - Positional stability
- ITTC (International Towing Tank Conference)
- Standard maneuvers
  - Turning circle
  - Kempf's ZigZag maneuver
  - Pull-out
  - Dieudonné's Spiral
  - Stopping trials

Turning circle

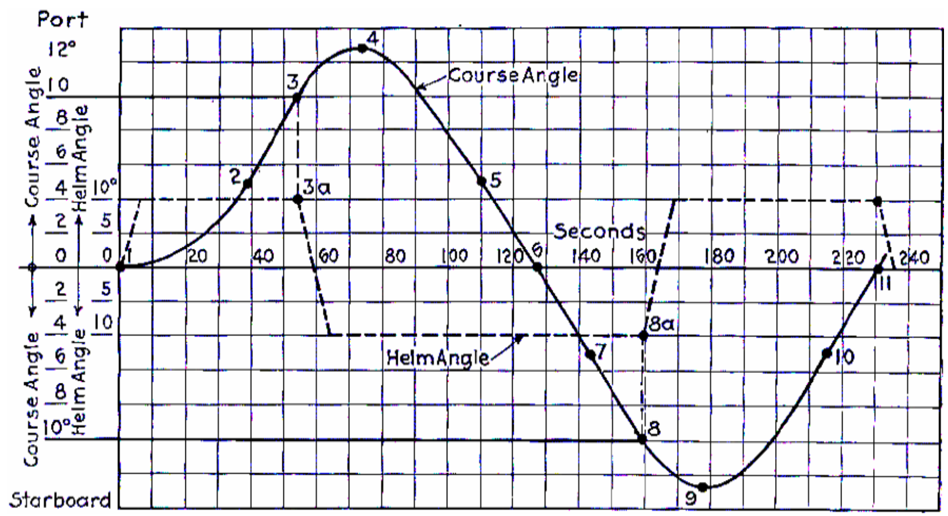
- Simple maneuver for standard ships with rudder
- Standard measures of maneuverability in course changing
  - Steady turning radius
  - Advance
  - Tactical diameter
  - Transfer



in Thor Fossen, “Handbook of Marine Craft Hydrodynamics and Motion Control”, Wiley

ZigZag Maneuver

- The maneuver determines several characteristics of the yaw response:
  - Response time to reach a given heading
  - Yaw overshoot (amount exceeding the setting 20º on rudder reversal)
  - Total period for 20º oscillations



Original figure in [1]

1. With zero rudder achieve steady speed for 1 min
2. Set the rudder to 20º e keep it ther until the ship turns 20º
3. Set the rudder to -20º and hold it until the vessel turns -20º with respect to the initial heading
4. Repeat

[1] G. Kempf, “Measurements of the Propulsive and Structural Characteristics of Ships”, SNAME Transactions, Vol 40. , 1932

Dieudonné Spiral

- Open water maneuver for ships

1. Achieve steady speed and direction for 1 min (no changes in setting speed after)
2. Turn rudder quickly by 15º and keep I there until steady yaw rate is maintained for 1 min
3. Reduce rudder by 5ª and keep it there until steady yaw rate is maintained for one minute
4. Repeat in decrements ot -5º to -15º
5. Return back up to 15º

- This open water maneuver puts the vessel in a growing spiral and then in a contracting spiral.

- The test reveals if the vessel has a hysteresis in the yaw rate r.
  - Ex: if for the first 15º deflection the ship turns right and that the yaw rate at zero rudder on the way back is still to the right, the ship is gotten stuck and requires negative rudder action to end the turn. The vessel is unstable.

Guidance

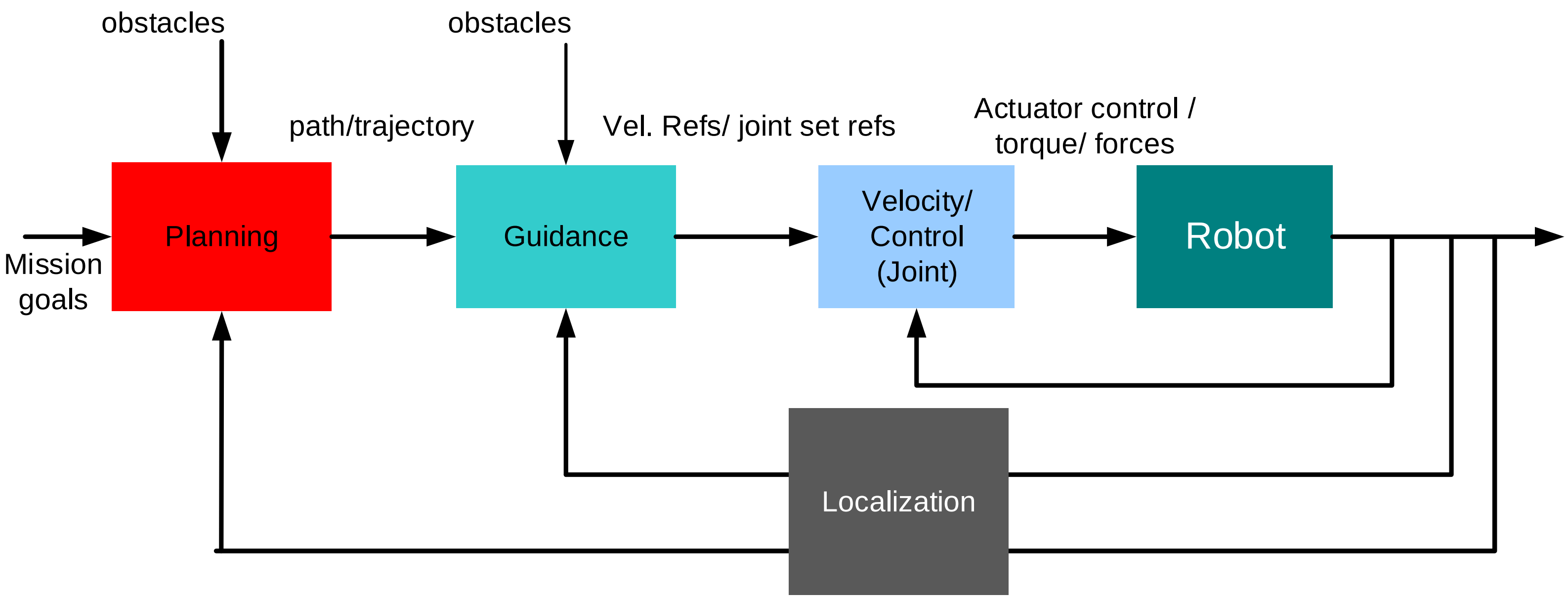
Control Problems

- Point stabilization/ regulation
  - Stabilize the vehicle
  - Ex: hovering, zeroing the error to a given position and orientation
  - Non-holonomic constraints – there is no continuous smooth feedback law

- Path following
  - The vehicle should follow a path in space without temporal constraints
  - Velocity control an independent problem (usually to a fixed value or profile)
  - Practical problem in many circumstances

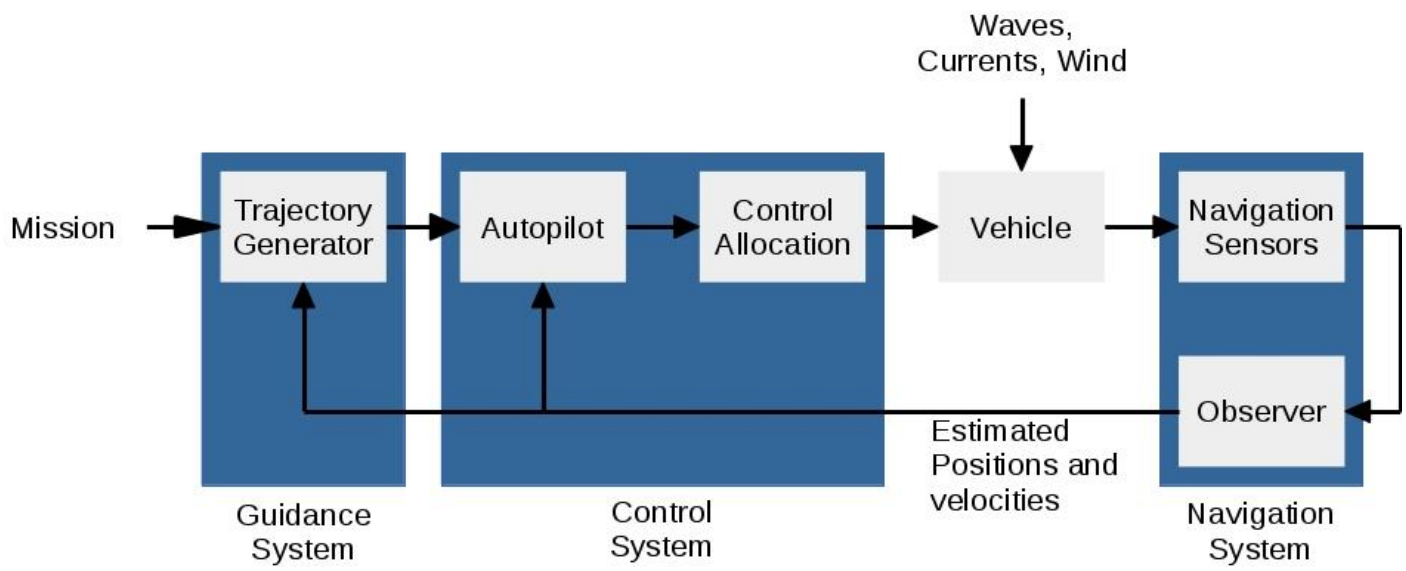
- Trajectory tracking
  - Tracking of a time parameterized sequence

Guidance


obstacles
obstacles
path/trajectory
Vel. Refs/ joint set refs
Actuator control / torque/ forces
Mission goals
Planning
Guidance
Velocity/ Control (Joint)
Robot
Localization

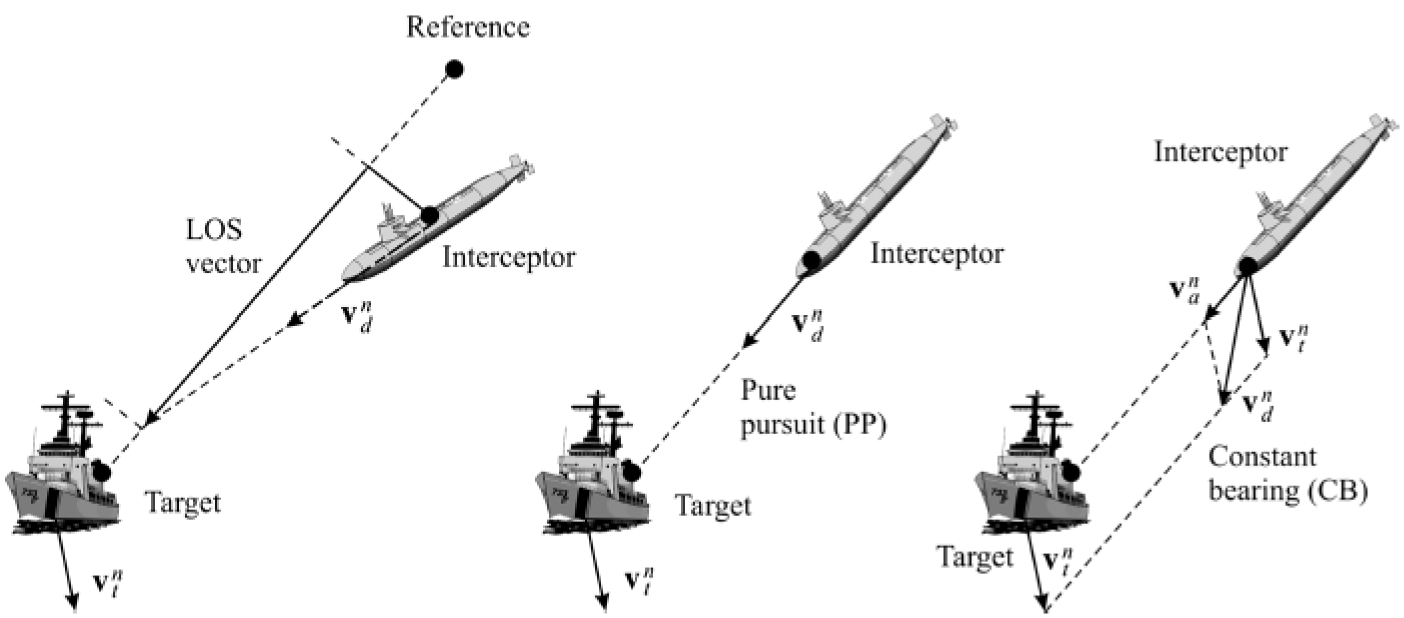
Guidance

- What should be my path?
- Definition of the path (waypoints, parameterized path, trajectory)
- Path following (or trajectory tracking control



Target Tracking

- Line-of-Sight (LOS) guidance
- Pure Pursuit (PP) guidance
- Constant Bearing (CB) guidance

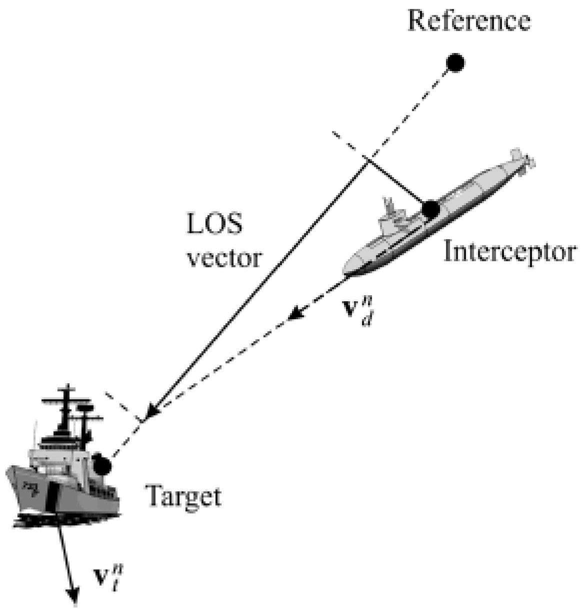


in Thor Fossen, "Handbook of Marine Craft Hydrodynamics and Motion Control", Wiley

Line of Sight

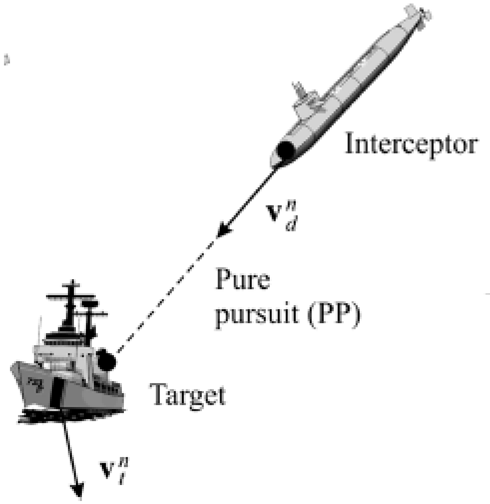
- A reference point and the target are used to define a LOS vector
  - In the case of waypoint following the reference point corresponds to the previous waypoint

- The vehicle points to the LOS vector, leading to a motion towards the reference-target line of sight



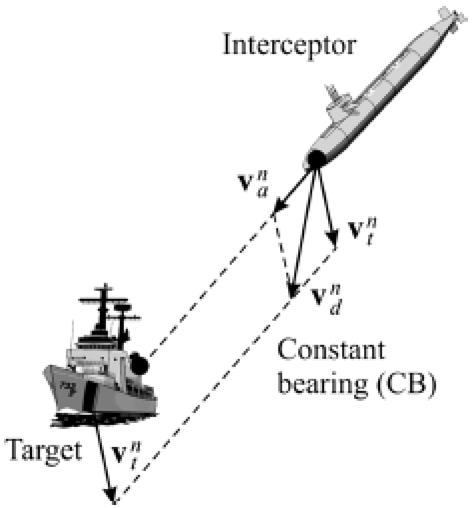
Pure Pursuit

- The vehicle points directly to the target (aligns the velocity with the vector between itself and the target
- Greedy approach



Constant

- Vehicle aligns its velocity along the LOS vector to the target reducing the LOS rotation rate as to perceived it as constant bearing
- Navigation in order to put the vehicle in a collision course (also used for obstacle avoidance)
- Common method is to make rotation rate of vehicle proportional to rotation rate of the LOS vector



Path generation

- Path generation can be made from a set of waypoints
- Common to have marine robot missions defined taking into account a waypoint database

- Waypoint set
  - Mission objectives
  - Environment conditions(winds, waves, currents)
  - Geographical information and morphology
  - Obstacles
  - Feasibility (limited by vehicle dynamics)

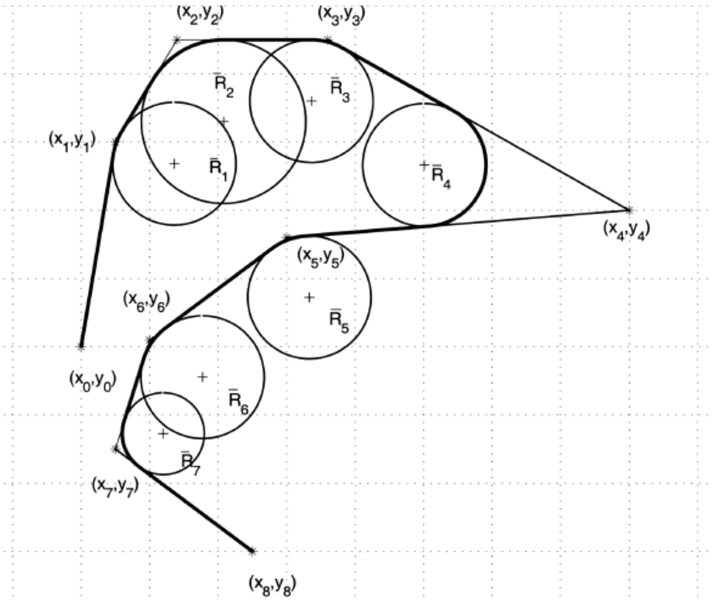
Waypoint Path generation

- Waypoint list is given by an ordered set of waypoints

$$wp_{list} = (x_0, y_0, z_0), (x_1, y_1, z_1)...(x_n, y_n, z_n)$$

- More information can be incorporated in the waypoints such as speed or orientation

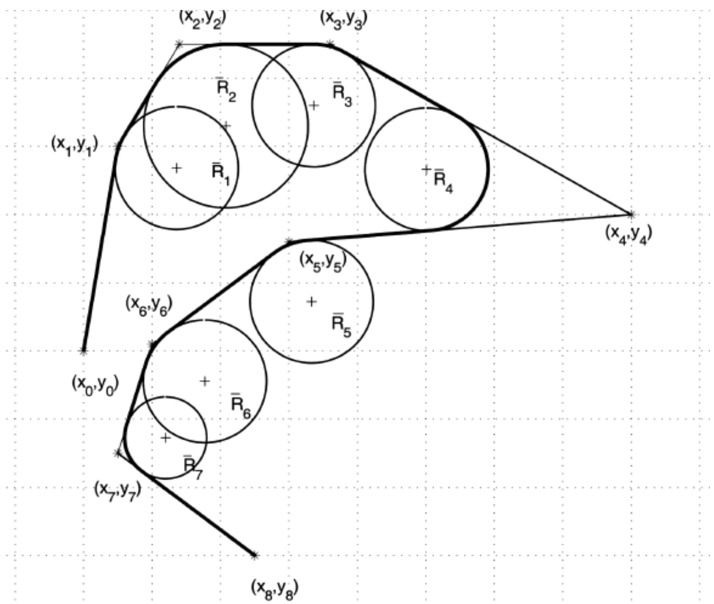
- Typical approach is to use straight lines and circular arcs



in Thor Fossen, "Handbook of Marine Craft Hydrodynamics and Motion Control", Wiley

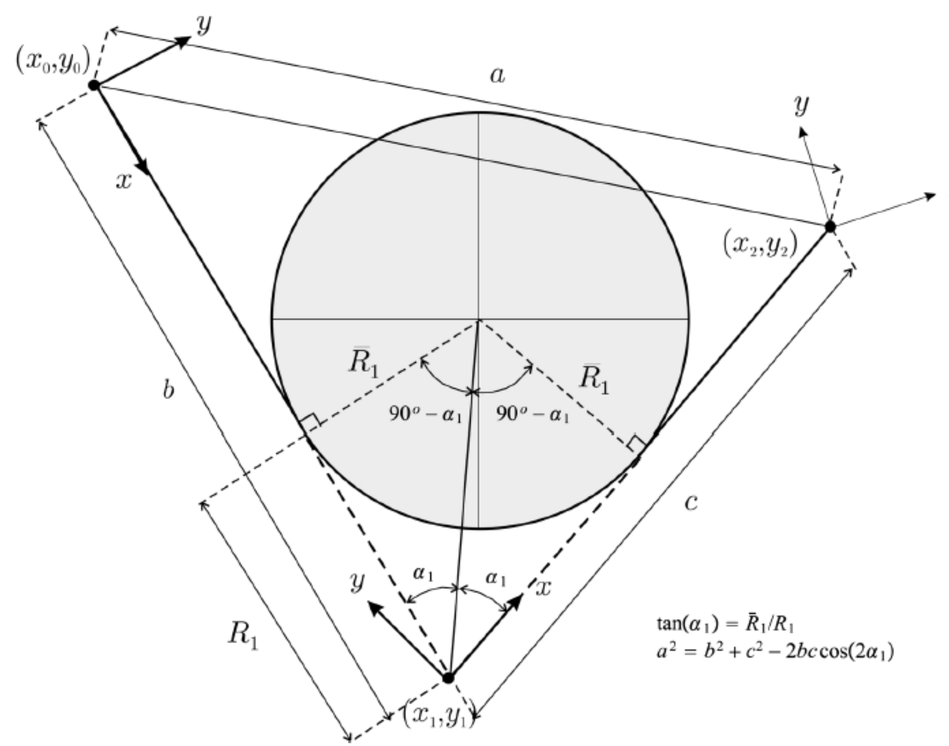
Waypoint path generation

- Common to represent desired path with straight lines and circle arcs
- The resulting trajectory is the one giving minimum path, however there are jumps in rotation velocity (r) when passing (curvature not continuous)



Waypoint path generation

- Turning point on the tangent point
- Radius $\bar{R}_1$ a parameter



in Thor Fossen, "Handbook of Marine Craft Hydrodynamics and Motion Control", Wiley

Kinematic control

- Results developed for other types of robots can be applied
- Ex:
  - differential drive land robots vs marine vehicles in horizontal plane

Controls: $(v_l, v_r) \implies (v, \omega)$

Controls: $(n_1, n_2) \underset{?}{\implies} (v, \omega)$

With no slideslip , one can assume n1,n2 common mode generating X, and differential mode acting on N

Kinematic control – example

- Kinematic model (inertial frame)

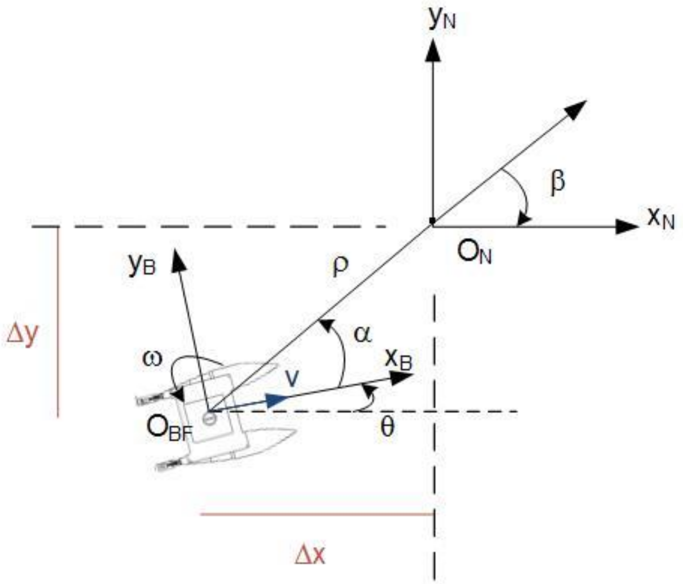
$$^{I}\begin{bmatrix}\dot{x}\\ \dot{y}\\ \dot{\theta}\end{bmatrix} = \begin{bmatrix}\cos\theta & 0\\ \sin\theta & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}v\\ \omega\end{bmatrix}$$

- Coordinate transformation, polar coordinates with origin in the final position

$$\rho = \sqrt{\Delta x^2 + \Delta y^2}$$

$$\alpha = -\theta + \operatorname{atan2}(\Delta y, \Delta x)$$

$$\beta = -\theta - \alpha$$



- System description in the new frame (polar coordinates)

$$\begin{bmatrix}\dot{\rho}\\ \dot{\alpha}\\ \dot{\beta}\end{bmatrix} = \begin{bmatrix}-\cos\alpha & 0\\ \frac{\sin\alpha}{\rho} & -1\\ -\frac{\sin\alpha}{\rho} & 0\end{bmatrix}\begin{bmatrix}v\\ \omega\end{bmatrix} \qquad \alpha \in I_1 \qquad I_1 = \left(-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

$$\begin{bmatrix}\dot{\rho}\\ \dot{\alpha}\\ \dot{\beta}\end{bmatrix} = \begin{bmatrix}\cos\alpha & 0\\ -\frac{\sin\alpha}{\rho} & 1\\ \frac{\sin\alpha}{\rho} & 0\end{bmatrix}\begin{bmatrix}v\\ \omega\end{bmatrix} \qquad v = -v \qquad \alpha \in I_2 \qquad I_2 = (-\pi, -\pi/2] \cup (\pi/2, \pi]$$

[1] A. Astolfi, "Exponential Stabilization of a Mobile Robot", in Proc. 3rd European Control Conference, Rome, 1995
[2] S. Lee et al., "A Stabile Target-Tracking Control for Unicycle Mobile Robots", IEEE IROS,, 2000

Kinematic control - example

- Coordinate transform **not defined for x=y=0**. In this point the transformation matrix Jacobian is not defined

- For $\alpha \in I_1$ robot points to the final point,
  for $\alpha \in I_2$ it's the opposite direction

- Defining the robot front in the initial position it is always possible to have for $t=0$. However there is no guarantee that α stays in I_1 for all t.

Kinematic control -example

- It is demonstrable that the for the linear control law

$$v = k_\rho \rho$$

$$\omega = k_\alpha \alpha + k_\beta \beta$$

Linear velocity proportional to distance error

Angular velocity proportional to orientation errors

- the closed loop system

$$\begin{bmatrix} \dot{\rho} \\ \dot{\alpha} \\ \dot{\beta} \end{bmatrix} = \begin{bmatrix} -k_\rho \rho \cos\alpha \\ k_\rho \sin\alpha - k_\alpha \alpha - k_\beta \beta \\ -k_\rho \sin\alpha \end{bmatrix}$$

- leads the robot to $(\rho, \alpha, \beta) = (0, 0, 0)$

- in the cartesian referential the control law originates non defined euations for $x=y=0$,

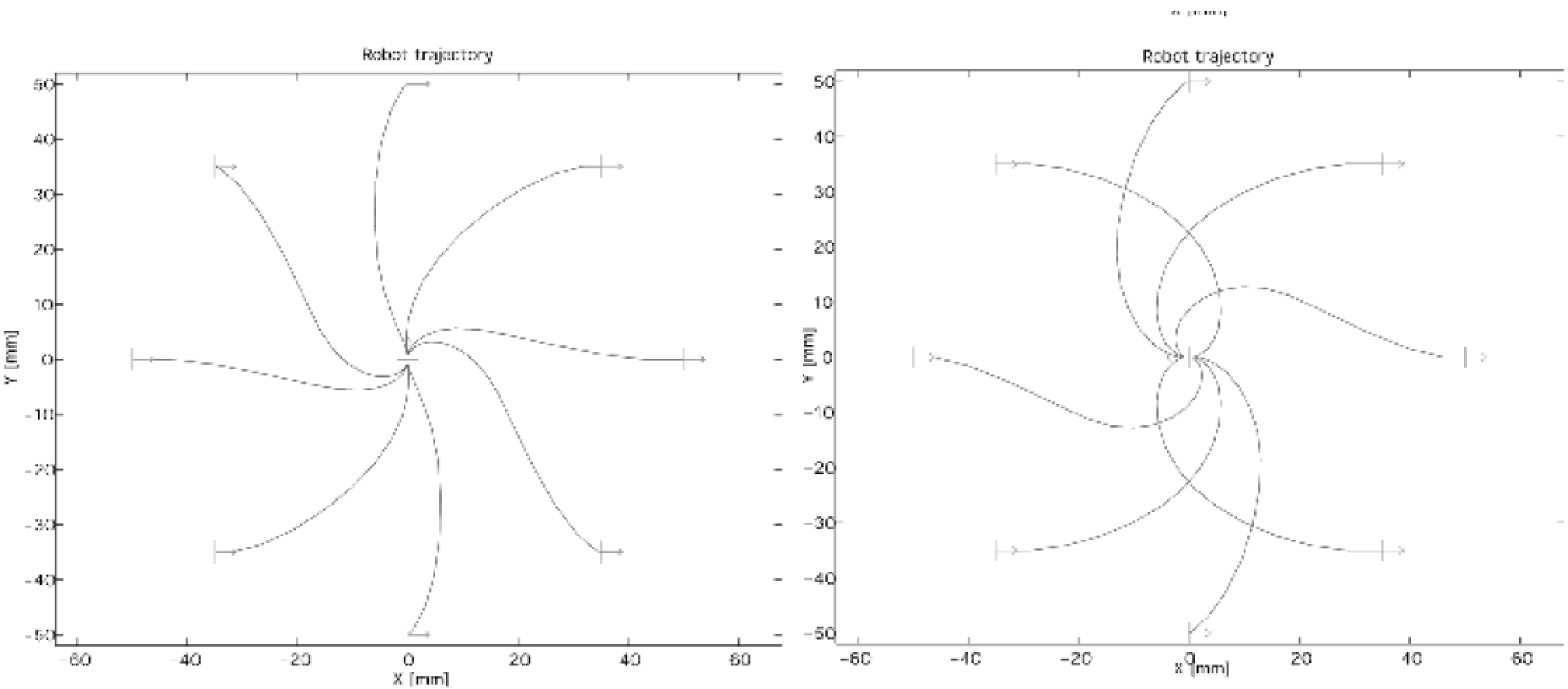
- angles α and β always between $(-\pi,\pi)$

- Control signal v has always constant signal
  - Motion direction positive or negative during the movement
  - Approach is performed in a “natural way” without direction reversal

Kinematic control – example

- Simulations for initial positions in the unitary circle

$$k = (k_\rho, k_\alpha, k_\beta) = (3, 8, -1.5)$$



Kinematic control - example

- The closed loop system is locally exponentially stable if

$$k_\rho > 0 \;; \quad -k_\beta > 0 \;; \quad k_\alpha - k_\rho > 0$$

- Proof:
  – for small values of x (linearization around equilibrium point),

$$\begin{bmatrix} \dot{\rho} \\ \dot{\alpha} \\ \dot{\beta} \end{bmatrix} = \begin{bmatrix} -k_\rho & 0 & 0 \\ 0 & -(k_\alpha - k_\rho) & -k_\beta \\ 0 & -k_\rho & 0 \end{bmatrix} \begin{bmatrix} \rho \\ \alpha \\ \beta \end{bmatrix} \qquad A = \begin{bmatrix} -k_\rho & 0 & 0 \\ 0 & -(k_\alpha - k_\rho) & -k_\beta \\ 0 & -k_\rho & 0 \end{bmatrix}$$

  – the system is locally exponentially stable if the eigenvalues of $\boldsymbol{A}$ have negative real part*

$$(\lambda + k_\rho)(\lambda^2 + \lambda(k_\alpha - k_\rho) - k_\rho k_\beta)$$

  – characteristic polynomial of $\boldsymbol{A}$ has all the roots with with negative real part for the stability condition

* Lyapunov linearization method, see nonlinear control references such as: “Applied nonlinear control”, Slotine and Li, Prentice-Hall

Kinematic control - example

- Condition that guarantees that the robot does not change direction

$$k_\rho > 0 \; ; \quad k_\beta < 0 \; ; \quad k_\alpha + \frac{5}{3}k_\beta - \frac{2}{\pi}k_\rho > 0$$

- Angle $\boldsymbol{\alpha(t)}$ belongs always to $\boldsymbol{I_1}$ or $\boldsymbol{I_2}$ for all $\boldsymbol{t}$, if $\boldsymbol{\alpha(0)}$ belongs to that intervals.

Path following

- Surface vehicle path following (Charlie USV) [1]

  – Kinematic model

$$
\begin{aligned}
\dot{x} &= U\cos\psi_e,\\
\dot{y} &= U\sin\psi_e,\\
\dot{\psi}_e &= r\left[\frac{u_r^2}{U^2} + \frac{u_r}{U^2}(\dot{x}_C\cos\psi + \dot{y}_C\sin\psi)\right] = r\eta(t)
\end{aligned}
$$

with

$$U = \sqrt{\dot{x}^2 + \dot{y}^2},$$ Vehicle velocity vector

$$\psi_e = \arctan\frac{\dot{y}}{\dot{x}}$$ Vehicle course

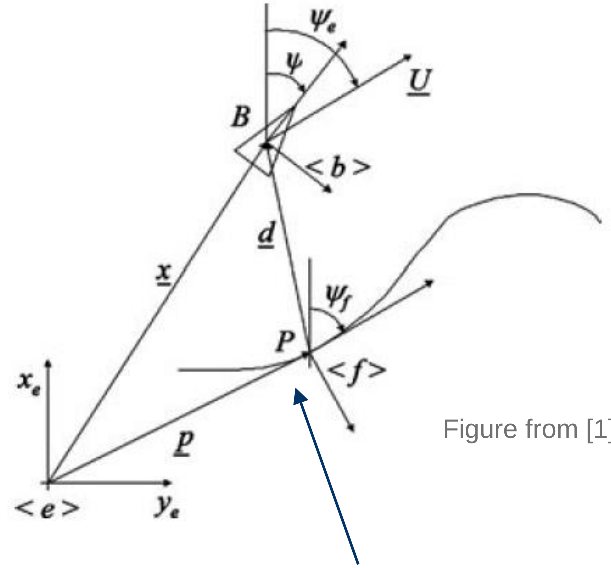


Figure from [1]

Serret-Frenet frame {f}

[1] M. Bibuli, “Path-Following Algorithms and Experiments for an Unmanned Surface Vehicle”, Journal of field Robotics, 2009

Path following

- Serret- Frenet frame {f} moves along the path to be followed – defines a virtual target

$\mathbf{p} = [x_P \;\; y_P \;\; 0]^T$ Position of arbitrary point p on the path

$\mathbf{x} = [x \;\; y \;\; 0]^T$ Vehicle position on {e}

$= [s_1 \;\; y_1 \;\; 0]^T$ Vehicle position on {f}

$\mathbf{R}_e^f(\psi_e)$ Rotation matrix from {e} to {f} (tangent at the path)

$$\mathbf{R}_e^f(\psi_e) = \begin{bmatrix} \cos\psi_f & \sin\psi_f & 0 \\ -\sin\psi_f & \cos\psi_f & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

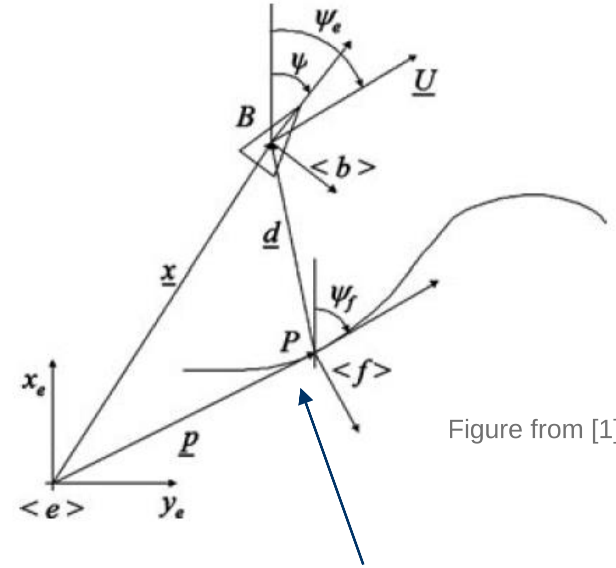


Figure from [1]

Serret-Frenet frame {f}

[1] M. Bibuli, “Path-Following Algorithms and Experiments for an Unmanned Surface Vehicle”, Journal of field Robotics, 2009

Path following

- Defining

$$r_f = \dot{\psi}_f$$ Rotation rate of {f}

$$s$$ Path variable (signed)

- the path curvature cc(s) and derivative

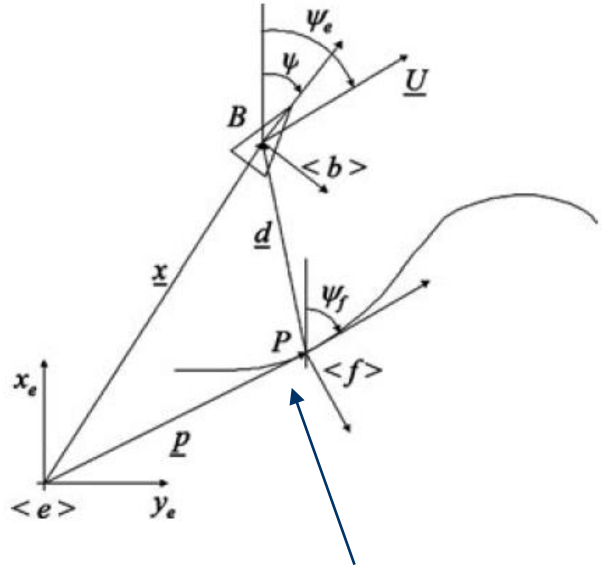
$$r_f = \dot{\psi}_f = c_c(s)\,\dot{s}$$

$$\dot{c}_c(s) = g_c(s)\,\dot{s}$$

- velocities of P and vehicle in {f} and {e}

$$\left(\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t}\right)_f = \begin{bmatrix} \dot{s} \\ 0 \\ 0 \end{bmatrix}$$

$$\left(\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t}\right)_e = \left(\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t}\right)_e + R^{-1}\left(\frac{\mathrm{d}\mathbf{d}}{\mathrm{d}t}\right)_f + R^{-1}([0 \quad 0 \quad r_f]^T \times \mathbf{d})$$



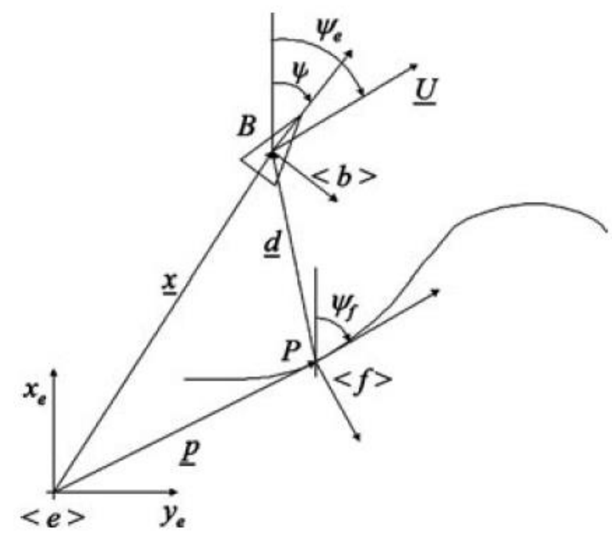
Serret-Frenet frame {f}

Path following

Solving for the serret-frenet velocities

$$\dot{s}_1 = [\cos\psi_f \quad \sin\psi_f] \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} - \dot{s}(1 - c_c y_1),$$

$$\dot{y}_1 = [-\sin\psi_f \quad \cos\psi_f] \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} - c_c \dot{s} s_1.$$



and substituting the kinematic model in these, one gets th kinematic model in the (s,y) coordinates (with $\beta = \psi_e - \psi_f$ the difference between course angle and tangential angle of {f}

$$\dot{s}_1 = -\dot{s}(1 - c_c y_1) + U\cos\beta,$$

$$\dot{y}_1 = -c_c \dot{s} s_1 + U\sin\beta,$$

$$\dot{\beta} = r_e - c_c \dot{s},$$

with

$$r_e = \dot{\psi}_e = r\eta(t)$$

Path following

- Path following controller decoupled from dynamics controller
  - Lower level “standard” controllers allow for yaw rate and surge speed control
  - Path follower provides references for yaw rate and surge

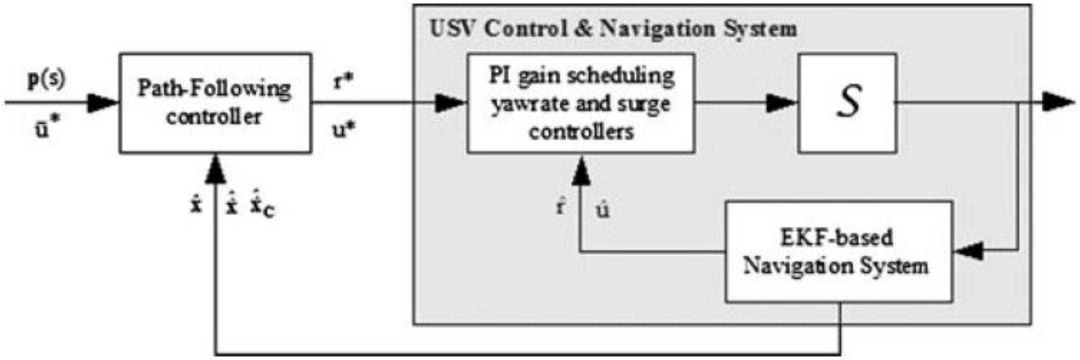


Figure from [1]

[1] M. Bibuli, “Path-Following Algorithms and Experiments for an Unmanned Surface Vehicle”, Journal of field Robotics, 2009

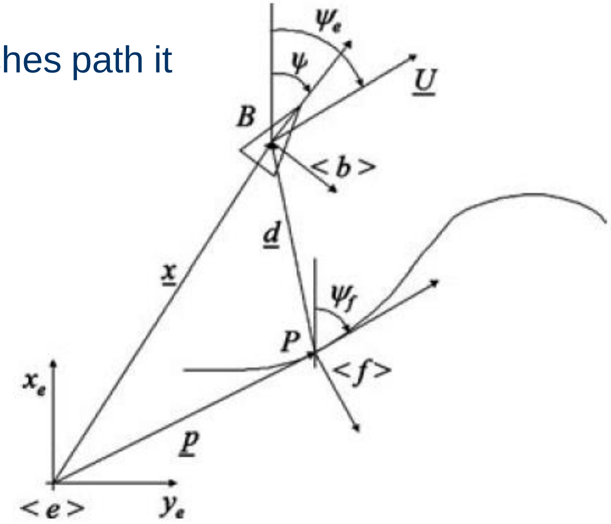
Path following

- Kinematic controller design
  - In general tracking controllers[1],[2] act on minimizing the distance d and error angle b, the approach taken in [3] is to choose P as moving along the path with a control law given an extra control design degree of freedom
  - Defines approach angle φ as function of tangent line (y1) to P given in frame {f}
  - $\varphi(y_1)$ must satisfy the requirement that once vehicle reaches path it stays there

$$\|\varphi(y_1)\| < \pi/2 \qquad y_1\varphi(y_1) \leq 0 \qquad \varphi(0) = 0$$

  - With $k_\varphi > 0$, $0 < \psi_a < \pi/2$

$$\varphi(y_1) = -\psi_a \tanh(k_\varphi y_1)$$



[1] Thor Fossen, "Handbook of Marine Craft Hydrodynamics and Motion Control", Wiley, 2011
[2] T. Fossen, M. Breivik, R. Skjetne "Line of sight path following of underactuated marine craft", 2003
[3] M. Bibuli, "Path-Following Algorithms and Experiments for an Unmanned Surface Vehicle", Journal of field Robotics, 2009

Path following

Approach angle β is imposed to track φ

Taking the CLF

$$V = \frac{1}{2}(\beta - \varphi)^2 \quad \Longrightarrow \quad \dot{V} = (\dot{\beta} - \dot{\varphi})(\beta - \varphi) = [r\eta(t) - c_c\,\dot{s} - \dot{\varphi}](\beta - \varphi)$$

The derivative of V is negative semi-definite by choosing the control law

$$r^* = \frac{1}{\eta(t)}[\dot{\varphi} - k_1(\beta - \varphi) + c_c(s)\,\dot{s}] \qquad k_1 \geq 0$$

(V is lower bounded and positive)

The motion on the path (for $\dot{V} = 0$) can be analyzed considering the CLF

$$V_E = \frac{1}{2}\left(s_1^2 + y_1^2\right) \quad \Longrightarrow \quad \dot{V}_E = (U\cos\beta - \dot{s})s_1 + U y_1 \sin\beta$$

As referred, the motion of the virtual target on the path can be chosen, by setting $\dot{s}$, the control law

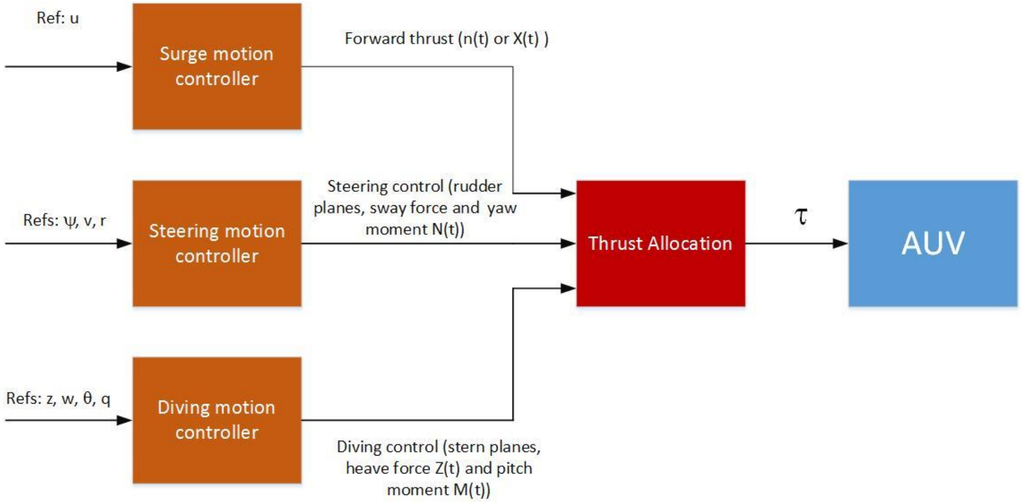
$$\dot{s}^* = U\cos\beta + k_2 s_1$$

Guarantees the derivative of VE as semi-negative[1].

Underwater vehicles

- Underwater vehicle motion in 3D for most applications allows for simplifying motion assumptions
  - Bathymetry and mapping missions performed in the horizontal plane
  - Terrain following missions
  - Oceanographic vertical profiling (or seesaw maneuvers) in the vertical plane (as for ex: for underwater gliders)
  - Or low speed motion as in inspection tasks with ROV or intervention type AUVs (iAUV)

- Decoupled motion control in surge, steering and dive modes
  - Seminal work of Healey demonstrated this approach to a highly actuated AUV



[1] A. Healey, D. Lienard, "Multivariable Sliding Mode Control for Autonomous Diving and Steering of Unmanned Underwater Vehicles", IEEE Journal of Oceanic Engineering, 1993