Model predictive control - Introduction

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7 June 2016

Outline

- Motivations and main ideas
- Ingredients and features of MPC regulators
- Model predictive control for tracking
- Model predictive control for linear systems and solution
- Remarks

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Some data from

A survey of industrial model predictive control technology

J. Qin, T. Badgwell
Control Engineering Practice, 11 (2003), pp. 733-764

Motivations and main ideas

Table 6
Summary of linear MPC applications by areas (estimates based on vendor survey; estimates do not include applications by companies who have licensed vendor technology)[a]

Area	Aspen Technology	Honeywell Hi-Spec	Adersa[b]	Invensys	SGS[c]	Total
Refining	1200	480	280	25		1985
Petrochemicals	450	80	—	20		550
Chemicals	100	20	3	21		144
Pulp and paper	18	50	—	—		68
Air & Gas	—	10	—	—		10
Utility	—	10	—	4		14
Mining/Metallurgy	8	6	7	16		37
Food Processing	—	—	41	10		51
Polymer	17	—	—	—		17
Furnaces	—	—	42	3		45
Aerospace/Defense	—	—	13	—		13
Automotive	—	—	7	—		7
Unclassified	40	40	1045	26	450	1601
Total	1833	696	1438	125	450	4542
First App.	DMC:1985	PCT:1984	IDCOM:1973			
	IDCOM-M:1987	RMPCT:1991	HIECON:1986	1984	1985	
	OPC:1987					
Largest App.	603 × 283	225 × 85	—	31 × 12	—	

[a] The numbers reflect a snapshot survey conducted in mid-1999 and should not be read as static. A recent update by one vendor showed 80% increase in the number of applications.

Some data from

Economic assessment of advanced process control – a survey and framework

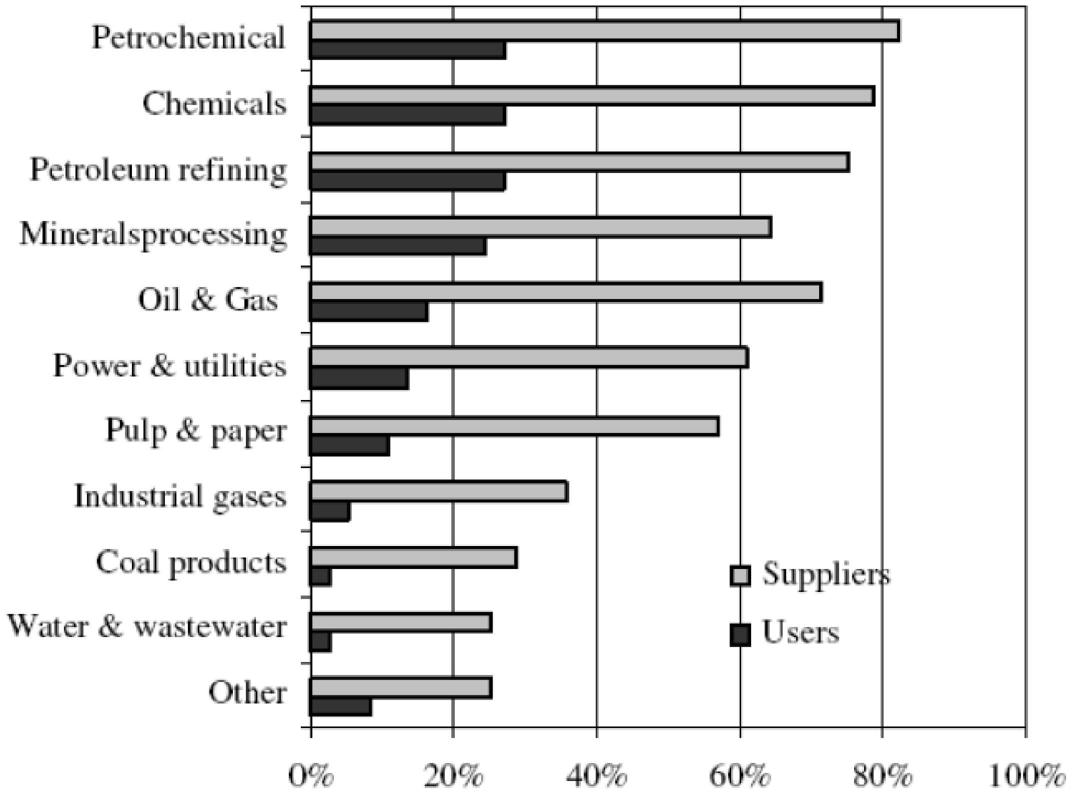
M. Bauer, I. K. Craig
Journal of process control, 18 (2008), pp. 2-18

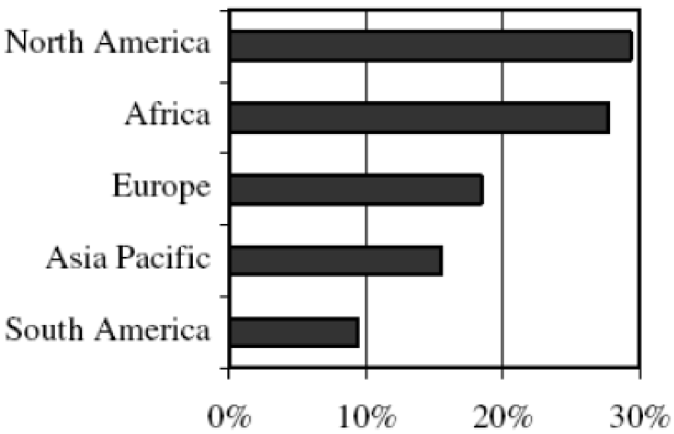
Motivations and main ideas

A key objective of industrial advanced process control (APC) projects is to stabilize the process operation. In order to justify the cost associated with the introduction of new APC technologies to a process, the benefits have to be quantified in economic terms.
In the past, economic assessment methods have been developed, that link the variation of key controlled process variables to economic performance quantities.
This paper reviews these methods and incorporates them in a framework for the economic evaluation of APC projects.
A web-based survey on the economic assessment of process control has been completed by over 60 industrial APC experts. The results give information about the state-of-the-art assessment of economic benefits of advanced process control

Motivations and main ideas

66 APC experts (38 APC users and 28 APC suppliers)





Motivations and main ideas

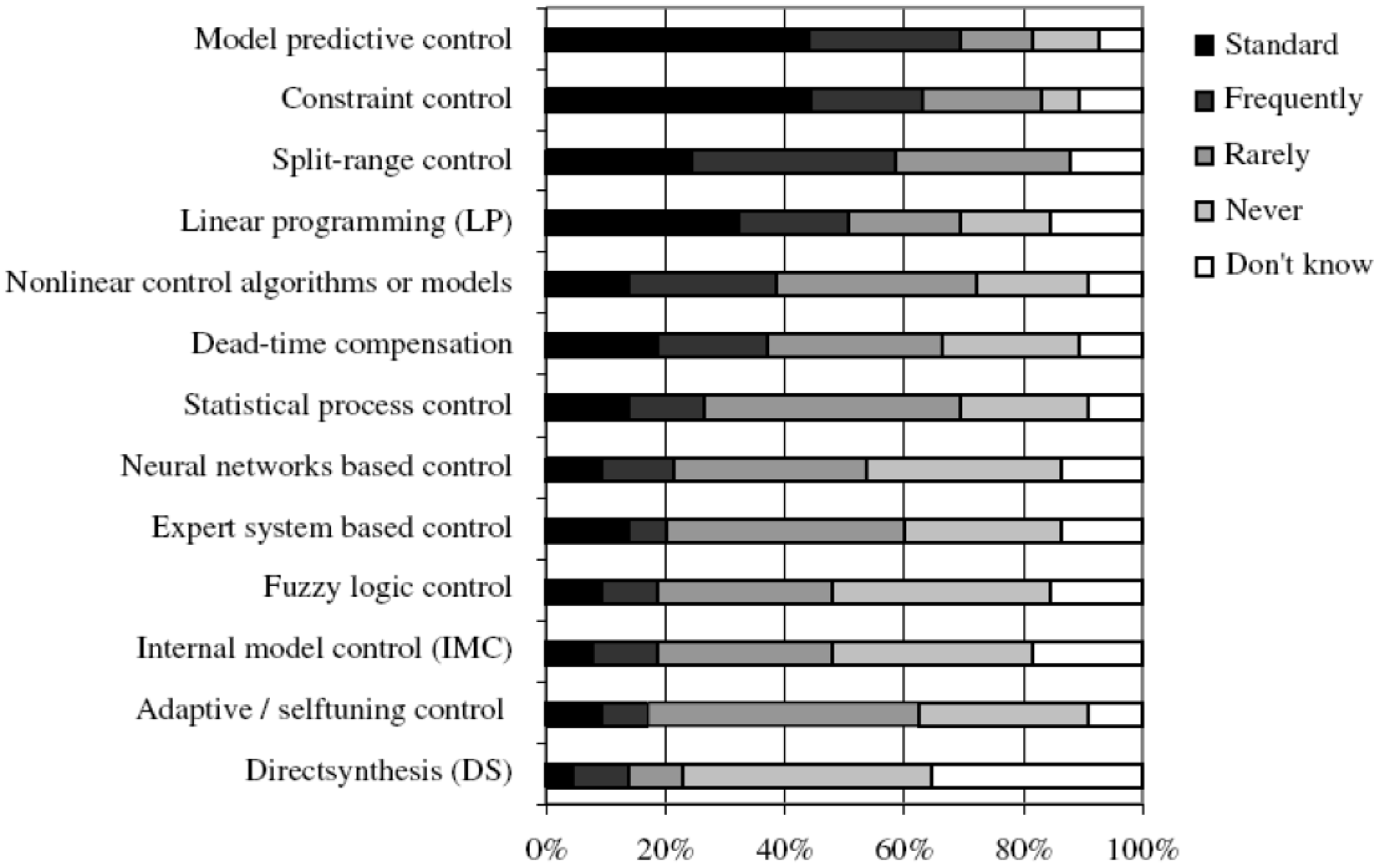


Fig. 4. Industrial use of APC methods: survey results.

Motivations and main ideas

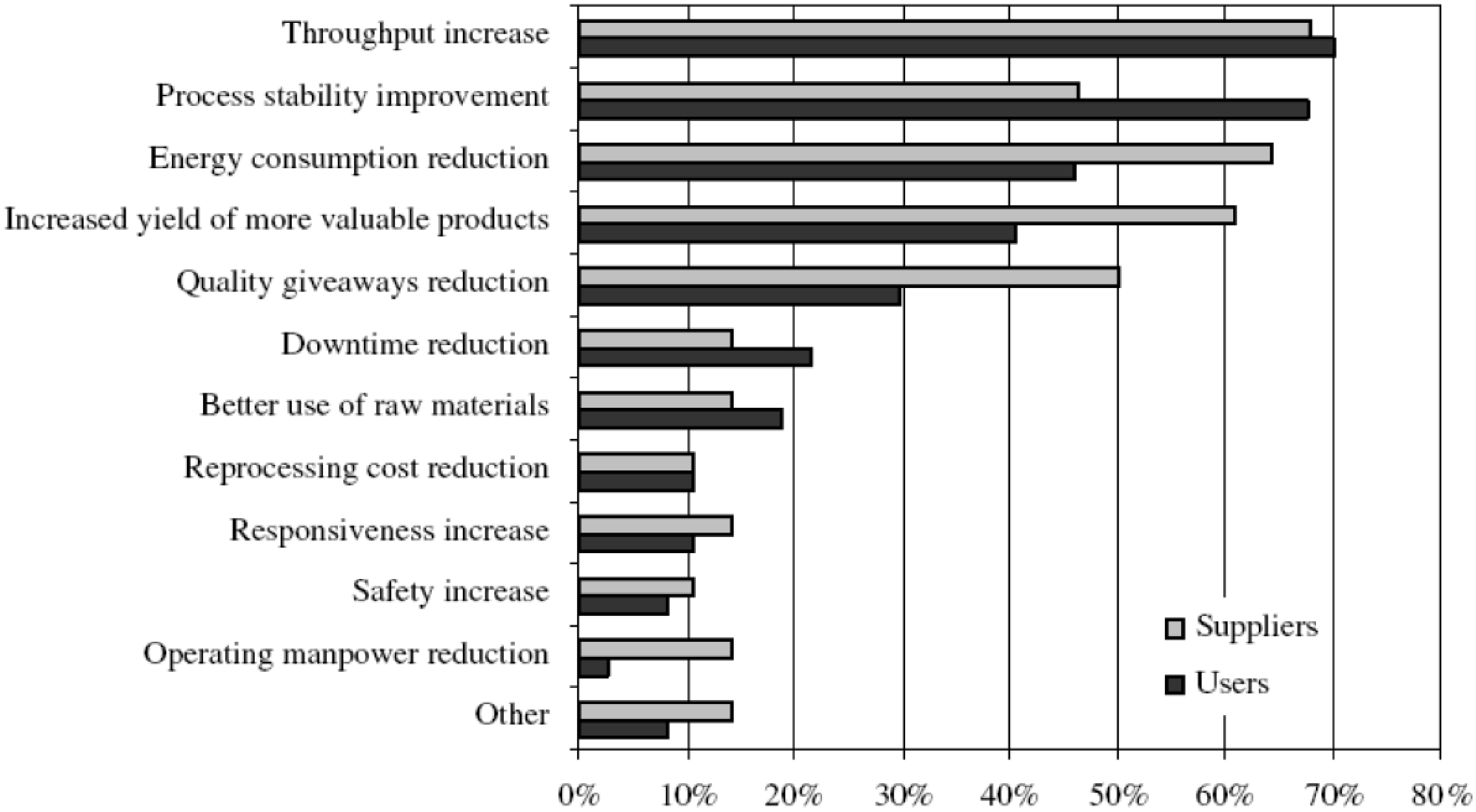
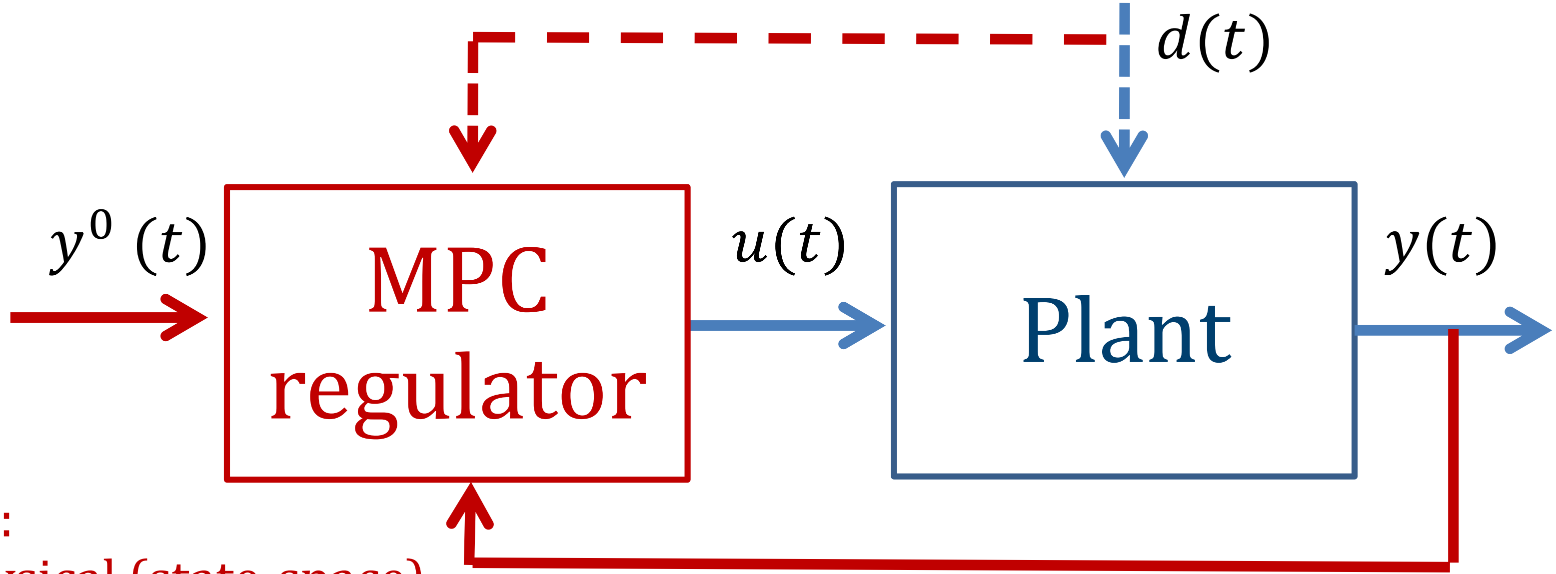


Fig. 6. Main profit contributors: survey results.

Motivations and main ideas



Model:

- Physical (state-space)
- Identified

Predictive:

- Actions are taken based on predictions
- Forecasts of disturbances and references are used

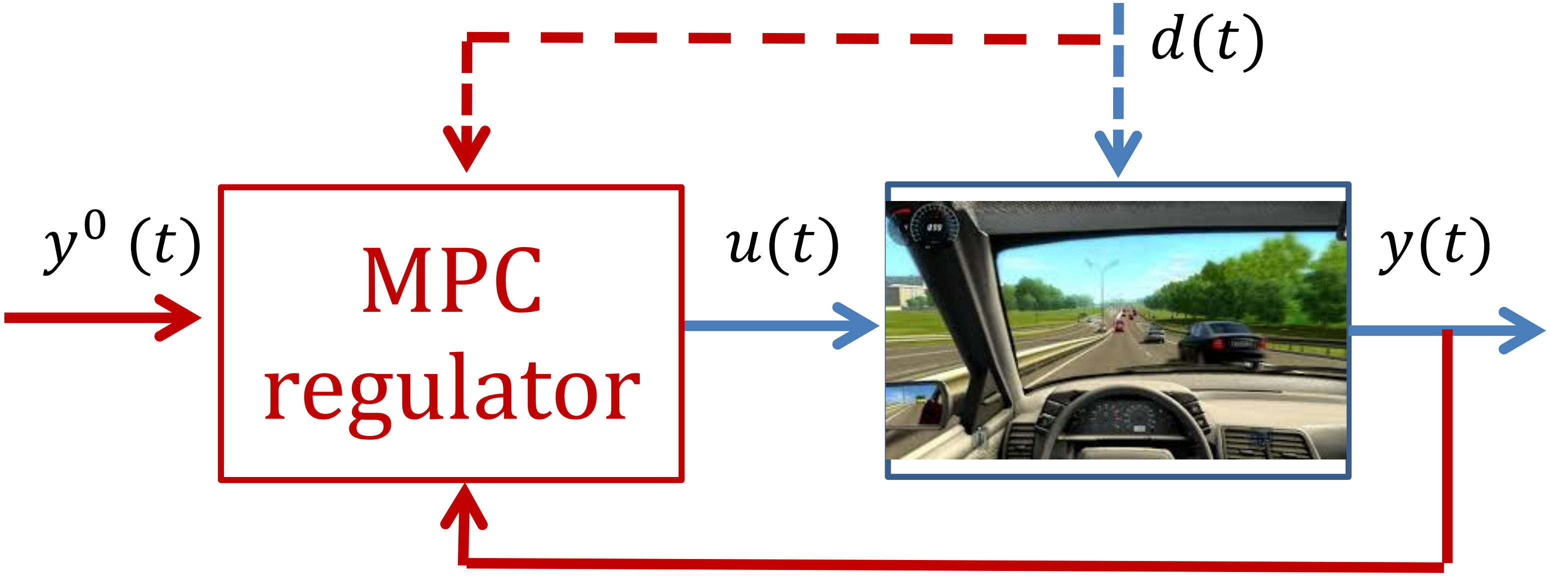
Control:

- Recursive solution to a finite horizon optimization problem
- State/input constraints are enforced

Plant:

- Complex
- Large scale
- Non linear
- Continuous/discrete time
- Perturbed

Motivations and main ideas



At time t

- Compute **optimal** future input sequence
  $u(t), u(t+1), \dots, u(t+N-1)$
  based on the current **model**, and on the **predictions** of references and disturbances.
- Apply $u(t)$

At time t+1

- Compute **optimal** future input sequence
  $u(t+1), u(t+2), \dots, u(t+N)$
  based on the current **model**, and on the **predictions** of references and disturbances.
- Apply $u(t+1)$

Receding horizon principle

Motivations and main ideas

Model predictive control is a family of algorithms that enables to:

- Include explicitly in the problem formulation contraints on input/state/output variables, and also logic relations
- Consider MIMO systems of relevant dimensions
- Optimize the system operation
- Use simple models for control (obtained, e.g., by identification tests) or very detailed nonlinear ones

Outline

- ➢ Motivations and main ideas
- ➢ Ingredients and features of MPC regulators
- ➢ Model predictive control for tracking
- ➢ Model predictive control for linear systems
- ➢ Remarks

Ingredients

1) The system model (discrete time):

$$x(t+1) = f(x(t), u(t))$$

- f is continuous
- $f(0,0)=0$ (i.e., 0 is an equilibrium point)

2) The constraints

Constraints are imposed on input and state variables

$$x(t) \in \mathbb{X}$$
$$u(t) \in \mathbb{U}$$

Suitable sets

- $\mathbb{X}$ closed
- $\mathbb{U}$ compact
- They contain the origin (at least for regulation)
- Better if convex

Ingredients

3) Cost function (regulation):

For practical (e.g., computational) reasons the cost is defined over a finite – prediction – horizon of length N steps.

At time t the following is minimized:

$$J = \sum_{k=t}^{t+N-1} l\big(x(k), u(k)\big) + V_f(x(t+N))$$

Stage cost:

- Continuous
- $l(0,0)=0$
- Definite positive

Terminal cost:

- Continuous
- $V_f(0)=0$
- Definite positive

Ingredients

The optimization problem (MPC)

At time t solve

$$\min_{u(t),\ldots,u(t+N-1)} J$$

Subject to

$x(t+1) = f(x(t), u(t))$	System dynamic model
$x(t) \in \mathbb{X}$	State constraints
$u(t) \in \mathbb{U}$	Input constraints
$x(t+N) \in \mathbb{X}_f$	**Terminal state constraints**

Ingredients

The two ingredients that still need to be defined (terminal cost $V_f(x(t+N))$ and terminal constraint $x(t+N) \in \mathbb{X}_f$) are necessary to guarantee two fundamental properties:

- Recursive feasibility
- Convergence/stability

What are these properties?

Recursive feasibility

If the MPC optimization problem has a solution at the initial time step 0, then a solution of MPC exists at each time step $t \geq 0$

Necessary for preventing from having no solution at a given time -> no control input would be defined!

Ingredients

Convergence

If a solution of the MPC optimization problem exists at each time instant t, then

$$\|x(t)\| \to 0 \text{ as } t \to +\infty$$

To ensure recursive feasibility and convergence/stability we need to define the 4th main ingredient.

4) Auxiliary control law

It is defined as a control law $u(t) = \kappa(x(t))$ such that the origin is an asymptotically stable equilibrium for the controlled system

$$x(t+1) = f(x(t), \kappa(x(t)))$$

Ingredients

5) Terminal constraint set $\mathbb{X}_f$

It is defined in such a way that, if $x(t) \in \mathbb{X}_f$, then

$$x(t+1) = f(x(t), \kappa(x(t))) \in \mathbb{X}_f$$

$$x(t) \in \mathbb{X}$$

$$u(t) = \kappa(x(t)) \in \mathbb{U}$$

It is a positively invariant set!

i.e., $\mathbb{X} \supseteq \mathbb{X}_f$

For instance, if $x(t) \in \mathbb{X}_f$, then we could apply the auxilary control law for all future time instants, and we obtain that,

- $x(t) \to 0$ as $t \to +\infty$
- the state and input constraints are always verified.

Ingredients

6) Terminal cost V_f

It is defined in such a way that «it decreases» if the auxiliary control law is applied.

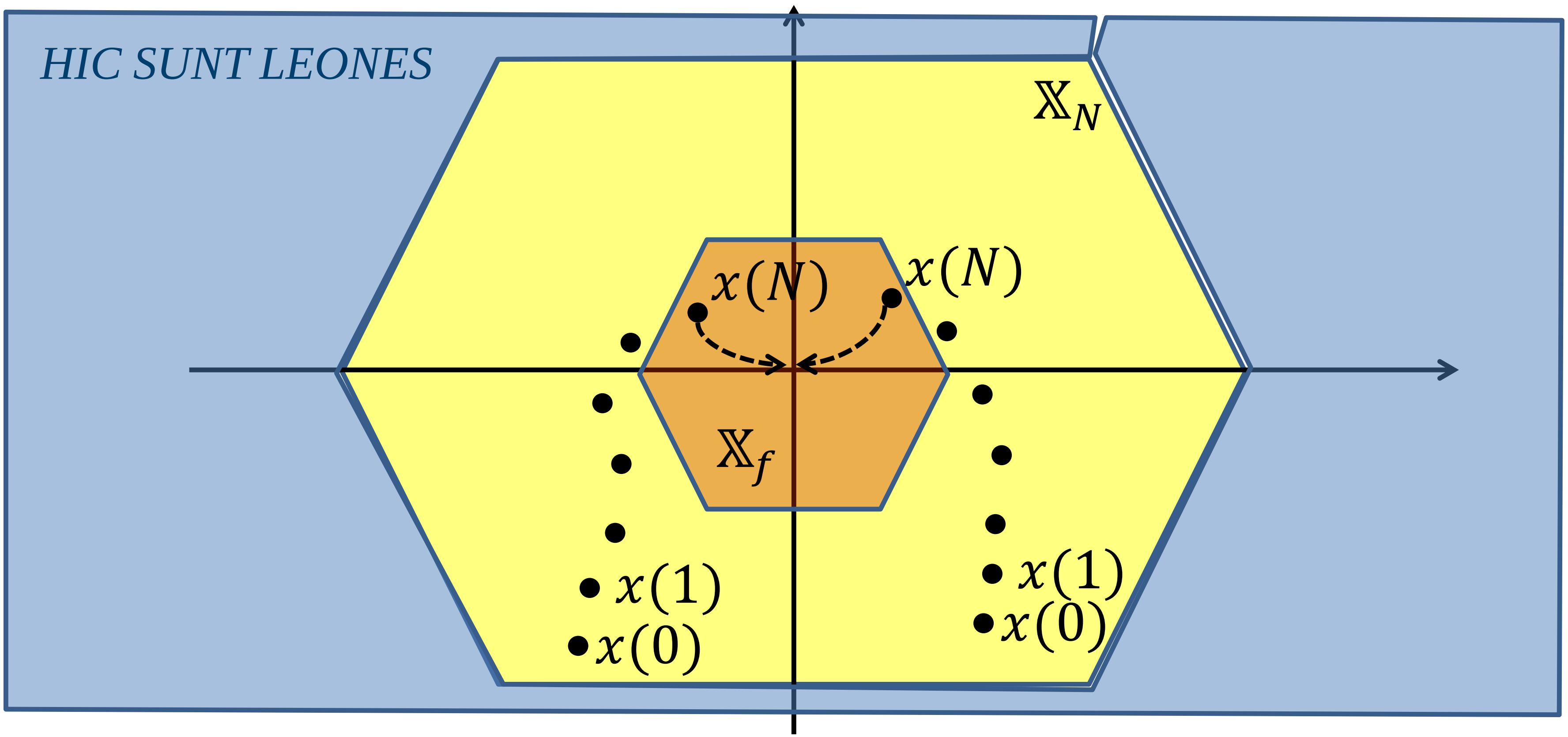
$$V_f\left(f\big(x,\kappa(x)\big)\right) \le V_f(x) - l(x,\kappa(x))$$

It is needed to establish formal convergence results

Ingredients

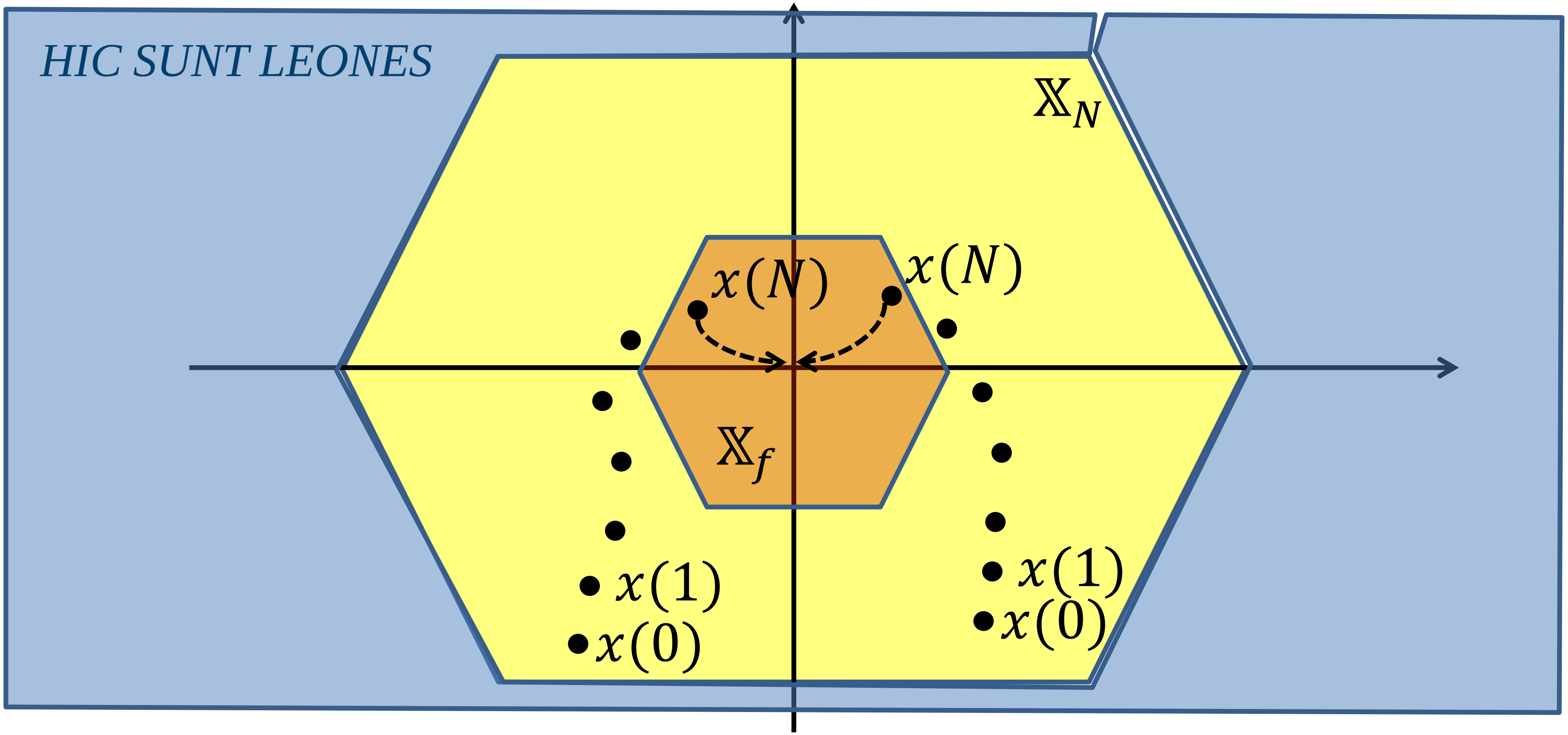
Basically, the latter ingredients are required for the following reason:

In N steps the terminal region must be reachable



This implicitly defines the set of all initial conditions such that the MPC optimization problem is feasible, $\mathbb{X}_N$

Ingredients



How to enlarge the terminal set?

- Increase the number of steps N (greater computational burden)
- In some applications (like autonomous vehicles) the **MPC for tracking** can be extremely beneficial

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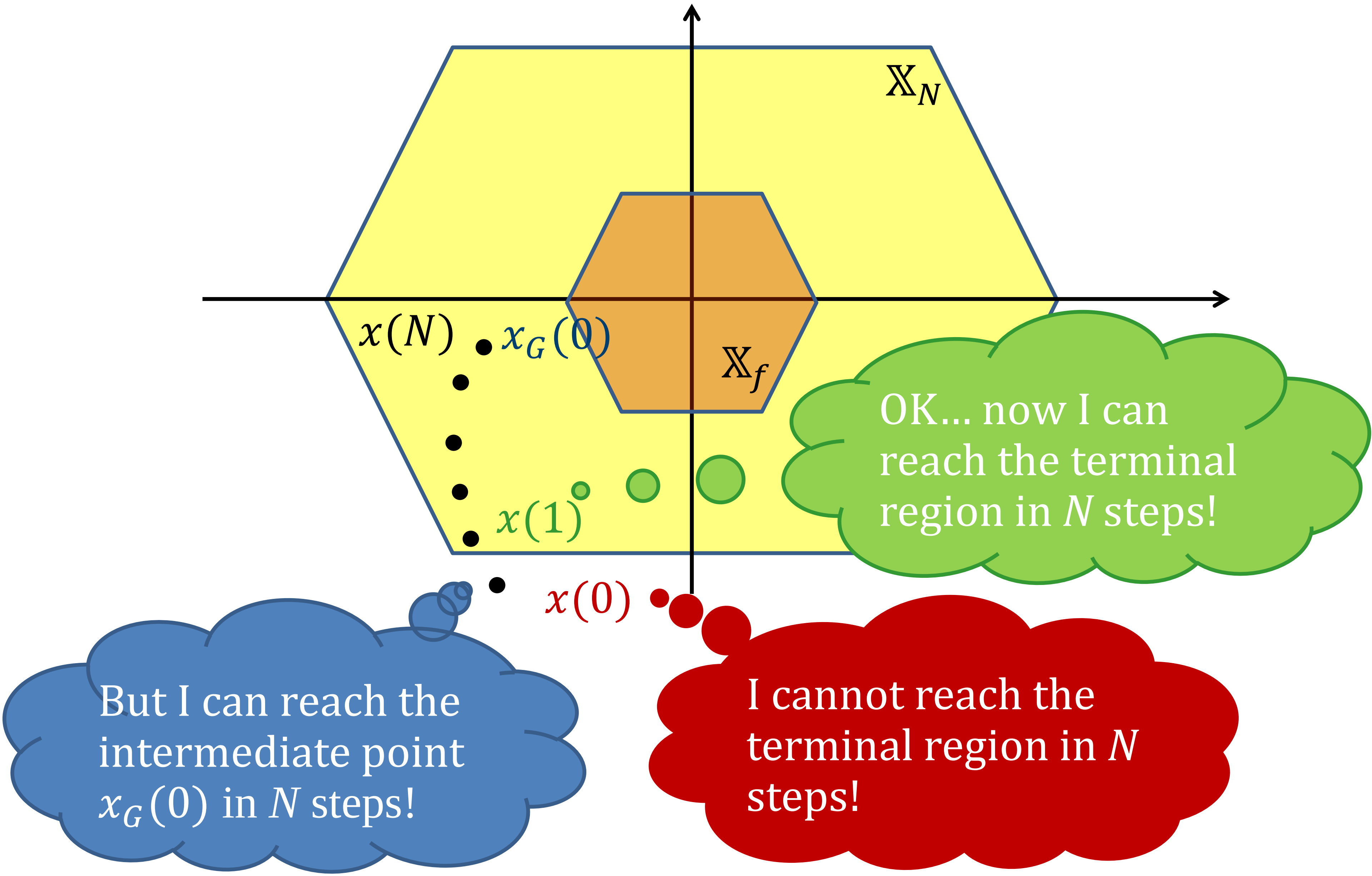
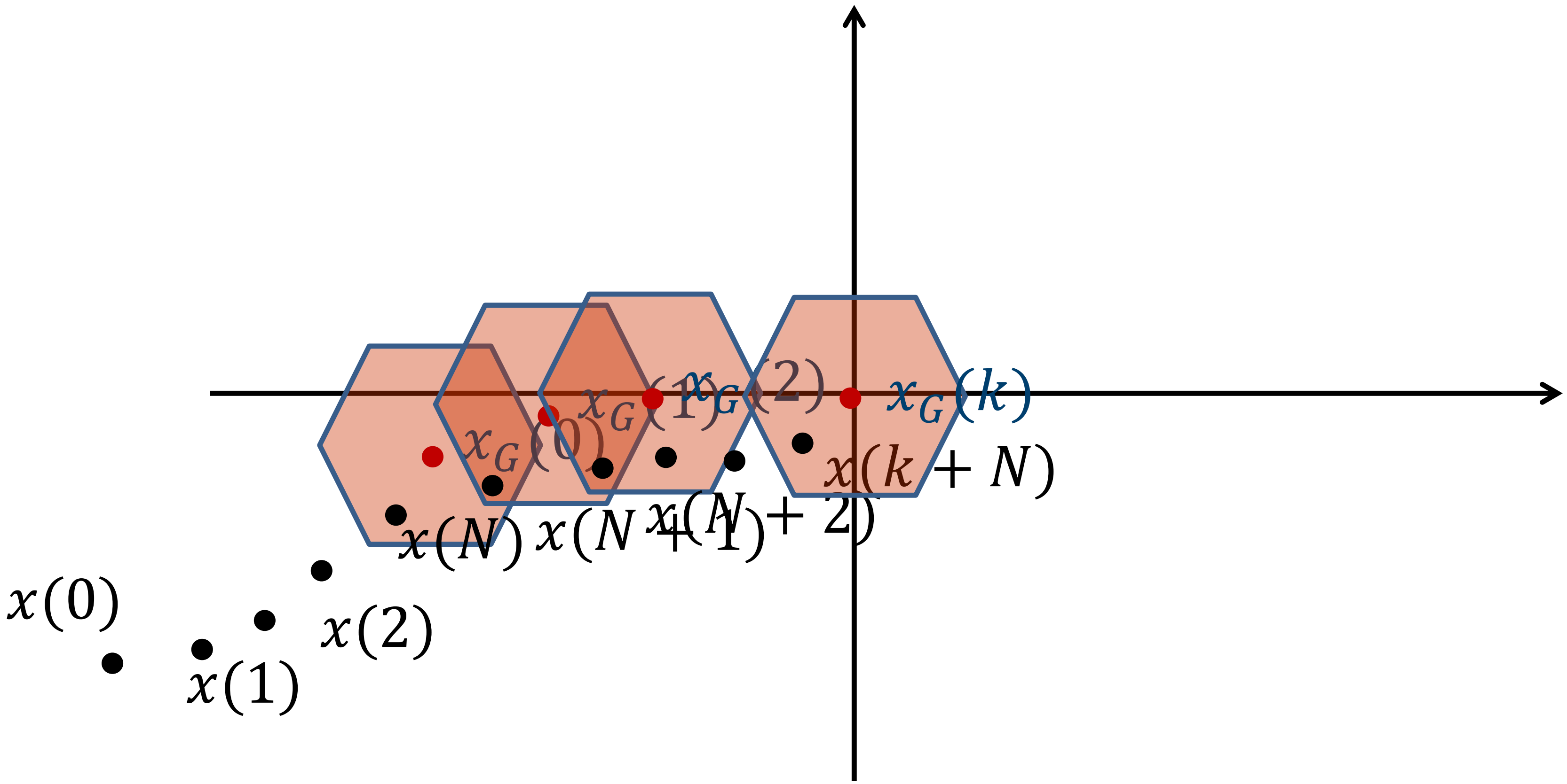
MPC for tracking – main idea



MPC for tracking – main idea



The solution lies in defining (in the optimization problem) intermediate goals $x_G(t)$, that can be reached in N steps, which eventually satisfy the property

$$x_G(t) \rightarrow \bar{x}_G \text{ (final goal) as } t \rightarrow +\infty$$

MPC for tracking - ingredients

1) The system model (discrete time):

$$x(t+1) = f(x(t), u(t))$$
$$y(t) = h(x(t), u(t))$$

- f is continuous
- $\bar{y}_G$ is our output goal
- $f(\bar{x}_G, \bar{u}_G)=\bar{x}_G$, i.e., $(\bar{x}_G, \bar{u}_G)$ is an equilibrium point and corresponds to the output $\bar{y}_G = h(\bar{x}_G, \bar{u}_G)$

2) The constraints

Constraints are imposed on input and state variables

$$x(t) \in \mathbb{X}$$
$$u(t) \in \mathbb{U}$$

→ Suitable sets

- $\mathbb{X}$ closed
- $\mathbb{U}$ compact
- They contain $\bar{x}_G$ and $\bar{u}_G$
- Convex

MPC for tracking – ingredients

3) Cost function:

At time t the following is minimized:

$$J = \sum_{k=t}^{t+N-1} l(x(k) - x_G(t), u(k) - u_G(t))$$

$$+V_f\big(x(t+N) - x_G(t)\big)$$

$$+\gamma \|y_G(t) - \bar{y}_G\|^2$$

Stage cost:

- Continuous
- $l(0,0)=0$
- Definite positive

Terminal cost:

- Continuous
- $V_f(0)=0$
- Definite positive

Cost on the temporary goal $y_G(t)$ (its deviation with respect to the final goal $\bar{y}_G$ is penalized)

$$f(x_G(t), u_G(t)) = x_G(t)$$
$$y_G(t) = h(x_G(t), u_G(t))$$

MPC for tracking - ingredients

The optimization problem (MPC)

At time t solve

$$\min_{u(t),\ldots,u(t+N-1),y_G(t)} J$$

Subject to

$$x(t+1) = f(x(t),u(t))$$
$$y(t) = h(x(t),u(t))$$

System dynamic model

$$x(t) \in \mathbb{X}$$

State constraints

$$u(t) \in \mathbb{U}$$

Input constraints

$$(x(t+N), y_G(t)) \in \mathbb{Z}_f$$

Terminal constraint

MPC for tracking - ingredients

The terminal cost V_f (same definition as before) and terminal constraint $(x(t+N), y_G(t)) \in \mathbb{Z}_f$) are necessary to guarantee:

- Recursive feasibility (same as before)
- Convergence/stability

Convergence

If a solution of the MPC optimization problem exists at each time instant t, then

$$\|y(t)\| \to \bar{y}_G \text{ as } t \to +\infty$$

4) Auxiliary control law

It is defined as a control law $u(t) = \kappa(x(t), y_{eq})$ such that the equilibrium point (x_{eq}, u_{eq}) is asymptotically stable for the controlled system $x(t+1) = f(x(t), \kappa(x(t), y_{eq}))$ and where

$$y_{eq} = h(x_{eq}, u_{eq})$$

MPC for tracking - ingredients

5) Terminal constraint

It is defined in such a way that, if $(x(t), y_{eq}) \in \mathbb{Z}_f$, then

$$(x(t+1), y_{eq}) \in \mathbb{Z}_f$$

$$\text{if } x(t+1) = f(x(t), \kappa(x(t), y_{eq}))$$

$$x(t) \in \mathbb{X}$$

$$u(t) = \kappa(x(t), y_{eq}) \in \mathbb{U}$$

It is a positively invariant set!

For instance, if$(x(t), y_{eq}) \in \mathbb{Z}_f$, then we could apply the auxilary control law for all future time instants, and we obtain that,

- $y(t) \rightarrow y_{eq}$ as $t \rightarrow +\infty$
- the state and input constraints are always verified.

MPC for tracking - ingredients

Terminal constraint - remark

The following solution can be applied: **terminal point as the terminal constraint set.**

$$(x(t+N), y_G(t)) \in \mathbb{Z}_f$$ **Terminal set**

⇓

$$x(t+N) = x_G(t)$$ **Terminal point**

In this case we don't need to define the auxiliary control law and the terminal set, which are rather complex to obtain, especially for non linear and/or large-scale systems.

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Linear systems - Ingredients

For simplicity we go back to the *regulation* case

1) The system model:

$$x(t+1) = Ax(t) + Bu(t)$$

The pair (A, B) must be controllable (at least stabilizable)

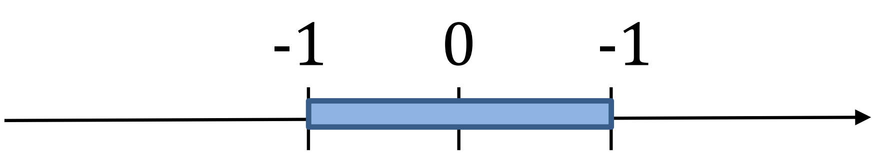
2) The constraints

Linear inequality constraints

$$x(t) \in \mathbb{X} \quad \Rightarrow \quad A_x x(t) \leq b_x$$

$$u(t) \in \mathbb{U} \quad \Rightarrow \quad A_u u(t) \leq b_u$$

For example, the *saturation* constraint $|u| \leq 1$ can be written as $\begin{bmatrix} -1 \\ 1 \end{bmatrix} u \leq \begin{bmatrix} 1 \\ 1 \end{bmatrix}$



Linear systems - Ingredients

3) Cost function – common choice:

At time t the following is minimized:

$$J = \sum_{k=t}^{t+N-1} \|x(k)\|_Q^2 + \|u(k)\|_R^2 + \|x(t+N)\|_P^2$$

Quadratic stage cost:

$$l\big(x(k), u(k)\big) = \|x(k)\|_Q^2 + \|u(k)\|_R^2 = x(k)^T Q x(k) + u(k)^T R u(k)$$

Where the symmetric matrices $Q \geq 0$ and $R > 0$ are arbitrary.

Quadratic terminal cost:

$$V_f\big(x(t+N)\big) = \|x(t+N)\|_P^2 = x(t+N)^T P x(t+N)$$

Where the symmetric matrix $P > 0$ is not arbitrary (see later).

Linear systems - Ingredients

4) Auxiliary control law

It is defined as a linear control law $u(t) = Kx(t)$ such that the matrix $F=A+BK$ is asymptotically stable, i.e., such that the system

$$x(t+1) = Ax(t) + BKx(t)$$

enjoys stability properties.

5) Terminal constraint set $\mathbb{X}_f$

In the linear case the terminal constraint can be enforced using linear inequalities

$$A_f x(t+N) \leq b_f$$

6) Terminal cost V_f

We can define $V_f\big(x(t+N)\big) = \|x(t+N)\|_P^2$, where P is the solution to the *Lyapunov inequality*:

$$(A+BK)^T P(A+BK) - P \leq -(Q + K^T RK)$$

Linear systems – MPC problem

The resulting MPC optimization problem can be cast as a QP problem (**quadratic program**)

⇨ Easy to be solved by available solvers (Matlab `quadprog`, IBM C-Plex `cplexqp`)

To see how, we define

$$X_t = \begin{bmatrix} x(t) \\ \vdots \\ x(t+N) \end{bmatrix}, U_t = \begin{bmatrix} u(t) \\ \vdots \\ u(t+N-1) \end{bmatrix}$$

And we compute that

$$X_t = \underbrace{\begin{bmatrix} I \\ \vdots \\ A^N \end{bmatrix}}_{\mathcal{A}} x(t) + \underbrace{\begin{bmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ A^{N-1}B & \cdots & B \end{bmatrix}}_{\mathcal{B}} U_t$$

Linear systems – MPC problem

At time t the **cost function** is:

$$J = \sum_{k=t}^{t+N-1} \|x(k)\|_Q^2 + \|u(k)\|_R^2 + \|x(t+N)\|_P^2$$

$$= \|X_t\|_{\mathcal{Q}}^2 + \|U_t\|_{\mathcal{R}}^2$$

where

$$\mathcal{Q} = \begin{bmatrix} Q & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & P \end{bmatrix} \quad \mathcal{R} = \begin{bmatrix} R & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & R \end{bmatrix}$$

We compute:

datum

Optimization variable

$$J = \|\mathcal{A}x(t) + \mathcal{B}U_t\|_{\mathcal{Q}}^2 + \|U_t\|_{\mathcal{R}}^2$$

$$= U_t^T(\mathcal{R} + \mathcal{B}^T\mathcal{Q}\mathcal{B})U_t + 2x(t)^T\mathcal{A}^T\mathcal{Q}\mathcal{B}U_t + x(t)^T\mathcal{A}^T\mathcal{Q}\mathcal{A}x(t)$$

Independent of the optimization variable: it can be neglected

Linear systems – MPC problem

At time t the **constraints** are:

$$\begin{bmatrix} A_u & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & A_u \end{bmatrix} \begin{bmatrix} u(t) \\ \vdots \\ u(t+N-1) \end{bmatrix} = \mathcal{A}_u U_t \leq \begin{bmatrix} b_u \\ \vdots \\ b_u \end{bmatrix} = \mathscr{b}_u$$

$$\begin{bmatrix} A_x & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & A_f \end{bmatrix} \begin{bmatrix} x(t) \\ \vdots \\ x(t+N) \end{bmatrix} = \mathcal{A}_x(\mathcal{A}x(t) + \mathcal{B}U_t) \leq \begin{bmatrix} b_x \\ \vdots \\ b_f \end{bmatrix} = \mathscr{b}_x$$

Overall:

$$\underbrace{\begin{bmatrix} \mathcal{A}_u \\ \mathcal{A}_x \mathcal{B} \end{bmatrix}}_{\mathcal{A}_{in}} U_t \leq \underbrace{\begin{bmatrix} \mathscr{b}_u \\ \mathscr{b}_x - \mathcal{A}_x \mathcal{A}x(t) \end{bmatrix}}_{\mathscr{b}_{in}(t)}$$

Linear systems – MPC problem

The resulting **quadratic program** is

$$\min_{U_t} U_t^T (\mathcal{R} + \mathcal{B}^T \mathcal{Q} \mathcal{B}) U_t + 2 x(t)^T \mathcal{A}^T \mathcal{Q} \mathcal{B} U_t$$

Subject to $\mathcal{A}_{in} U_t \leq \mathcal{b}_{in}(t)$

To solve it we can use, e.g., the Matlab function `quadprog`:

```
X = QUADPROG(H,f,A,b)
```

U_t

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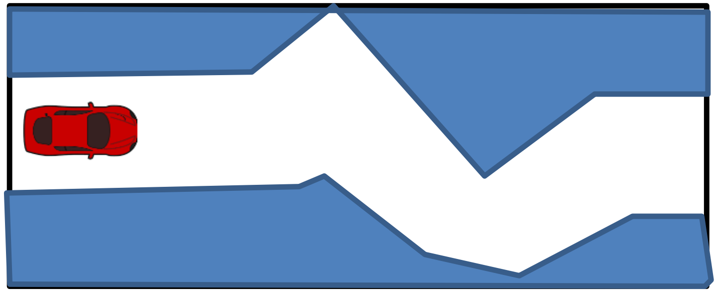
Remarks - advantages

- Possibility to account for **future** consequences of present actions:
  - Actions are taken based on predictions
  - Forecasts of disturbances and references are used

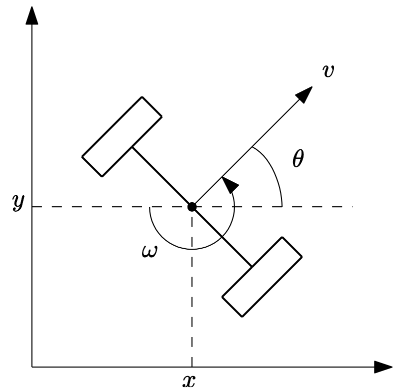
- We can naturally account for **constraints**:
  - On input variables, e.g., saturations
  - On «internal» state variables, e.g., velocities, accelerations
  - On «external» state variables, e.g., position, orientation, …



- We can chose the **optimal strategy** (in some sense) in each situation

Remarks - disadvantages

- Especially in the non linear case, MPC optimization problems are computationally burdensome!
- Solutions must be computed in a very short time interval (short sampling time)
- Vehicle models are frequently non linear, e.g., the (simple) unicycle model (differential drive)



$$\begin{cases} \dot{x} = \boxed{v} \cos\theta \\ \dot{y} = v \ \sin\theta \\ \dot{\theta} = \boxed{\omega} \end{cases}$$

inputs

Do we have to use a non linear MPC implementation? The MPC-based controller may be not applicable in practice…

Remarks – linearization for vehicle applications

We may resort to linearization techniques.
There are different possibilities:

1. **Linearization around a fixed – equilibrium – point** with $\theta = \bar{\theta}, v = 0, \omega = 0$.

$$\begin{cases} \dot{x} = v \cos\theta \\ \dot{y} = v \sin\theta \\ \dot{\theta} = \omega \end{cases} \Rightarrow \begin{bmatrix} \delta\dot{x} \\ \delta\dot{y} \\ \delta\dot{\theta} \end{bmatrix} = \begin{bmatrix} \cos\bar{\theta} & 0 \\ \sin\bar{\theta} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \delta v \\ \delta\omega \end{bmatrix}$$

➡ Not stabilizable!
Besides, the linearization error would be huge!

Remarks – linearization for vehicle applications

2. **Linearization around a trajectory**.

In the MPC framework, at time t the trajectory, predicted N steps forward at the previous time step (i.e., t-1) is available. We have the sequences

$$u(t-1|t-1), u(t|t-1), \dots, u(t+N-2|t-1)$$
$$x(t|t-1), \dots, x(t+N-1|t-1)$$

The linearized (affine) model, used for computing predictions at time t, is obtained linearizing the model in each point of the available predicted trajectory, i.e., for $k = t, \dots, t+N-1$

$$x(k+1) = A(k|t-1)\big(x(k) - x(k|t-1)\big) + B(k|t-1)\big(u(k) - u(k|t-1)\big) + x(k+1|t-1)$$

$$A(k|t-1) = \frac{\partial f}{\partial x}\bigg|_{u(k|t-1), x(k|t-1)} \quad B(k|t-1) = \frac{\partial f}{\partial u}\bigg|_{u(k|t-1), x(k|t-1)}$$

Remarks – linearization for vehicle applications

3. **Feedback linearization**

   Using a suitable internal control loop, we can make the system's dynamic evolve as a linear system.
   This approach is better analyzed in the following lecture.

➡ This approach may turn out to be prone to modelling errors.

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