Introduction to fuzzy logic

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Logic

Logic can be used to represent knowledge

Propositional logic: truth values for *propositions*

e.g.: *The grass is green* is *true*

First order predicate logic: truth values for *predicates*

There are variables and quantifiers

e.g.: ∃ X : (Green X) is *true*

Second order predicate logic: predicates about predicates

e.g.: ∃ X,Y : (Believe Y ((Green X) is *true)) is false*

Propositional logic

Propositional logics are concerned with *propositional* (or *sentential*) operators which may be applied to one or more propositions giving new propositions.

Accent on the truth value of propositions, not on their meaning, and on how these truth values are composed.

A logic is *truth functional* if the truth value of a compound sentence depends only on the truth values of the constituent atomic sentences, not on their meaning or structure. For such a logic the only important question about propositions is what truth values they may have.

In a classical, boolean or two-valued logic every proposition is either *true* or *false,* and no other feature of the proposition is relevant.

Example propositional logic

For example, the conjunction of the two sentences:

Grass is green

Pigs don't fly

is the sentence:

Grass is green and pigs don't fly

The conjunction of two sentences will be true if, and only if, each of the two sentences from which it was formed is true.

Other operators are disjunction (OR) and negation (NOT)

First order predicate logics

Same as propositional logics augmented with the possibility to define predicates on variables => existential and universal quantifiers (Exists, For All)

A predicate is a feature of language which you can use to make a statement about something, e.g., to attribute a property to that thing.

If you say "Peter is tall", then you have applied to Peter the predicate "is tall". We also might say that you have predicated tallness of Peter or attributed tallness to Peter.

Inference

In predicate logics it is possible to infer the truth value of a proposition by inferential mechanisms, such as modus ponens

E.g.: from

All men are mortal (modus ponens)

and

Socrates is a man (proposition)

we can infer that

Socrates is mortal (proposition)

Representational power of classical logic

Aristoteles already had some problems about the validity of “classical” logic as a knowledge representation tool.

For instance, how can we state the truth value of a proposition in the future?

E.g.: “Tomorrow it will rain”

Many-valued logics

We may introduce three-valued logics to cope with indecision:

true (1), false (0), and undefined (1/2)

So, why not introducing infinite-valued logics:

a continuum of truth values (e.g., in [0,1])

E.g., *Łukasievicz* (1930) logic L_1

- $T(\neg a) = 1 - T(a)$
- $T(a \wedge b) = \min(T(a), T(b))$
- $T(a \vee b) = \max(T(a), T(b))$
- $T(a \Rightarrow b) = \min(1, 1 + T(b) - T(a))$
- $T(a \Leftrightarrow b) = 1 - |T(a) - T(b)|$

Similarity and differences between L_1 and L_2

L_1 is isomorphic to the fuzzy set theory with standard operators as the classical logic L_2 is isomorphic to the set theory

Some tautologies valid in L_2 are no longer valid in L_1

- *Third excluded law* ($T(a \vee \neg a) = 1$)

  e.g., if the truth value of a is 0.7, then the truth value of ($a \vee \neg a$) is

  max{0.7, (1 - 0.7)} = 0.7

- *Non-contraddiction law* ($T(a \wedge \neg a) = 0$)

  e.g., if the truth value of a is 0.7, then the truth value of ($a \wedge \neg a$) is

  min{0.7, (1 - 0.7)} = 0.3

The liar paradox

“I’m a liar”

This sentence is a paradox in classical logic since no formula can have the same truth value of its negation

This may not be so in many-valued logics.

In Łukasiewicz logic, for instance, it can be that the truth value of φ is 0.5, and that of ¬φ is the same, so the proposition is consistent with the axioms, and it is no longer a paradox.

Fuzzy logic

Fuzzy logic is an infinite-valued logic, with truth values in [0,1]

Propositions are expressed as

A is L,

where:

- A is a linguistic variable
- L is a label denoting a fuzzy set

E.g., Temperature is HIGH

Linguistic variable

A linguistic variable is defined by a 5-tuple: (X, T(X), U, G, M)

X = name of the variabile

T(X) = set of terms for X (linguistic values), each corresponding to a fuzzy variable denoted by T(X) and ranging on U

U = universe of discourse defined on a base variable u

G = syntactic rule to generate the interpretation X of each value u

M = semantic rule to associate to X its meaning

E.g.: X is a linguistic variable labeled “Age”. U = [0, 100]. Terms for X are “old”, “middle-age”, “young”… The base variable u is the age in years. M is the definition in terms of fuzzy sets of the values of X. G is the fuzzy matching (interpretation) of u.

Simple propositions

$$\boldsymbol{p}\textbf{: }\boldsymbol{X}\textbf{ is L}$$

X is a linguistic variable

L denotes a fuzzy set F, defined on U, which represents a fuzzy predicate

$\mu_F(x)$ is the membership function defining F

$\mu_F(x)$ is interpreted as truth value of the proposition p

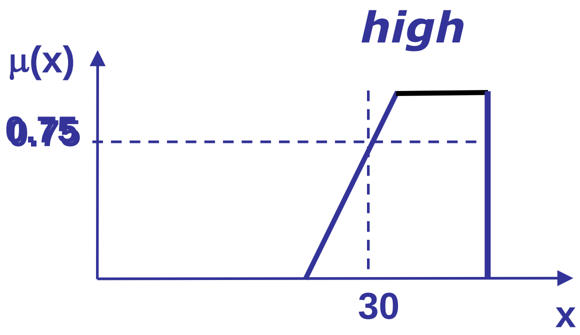
$$T(p) = \mu_F(x)$$

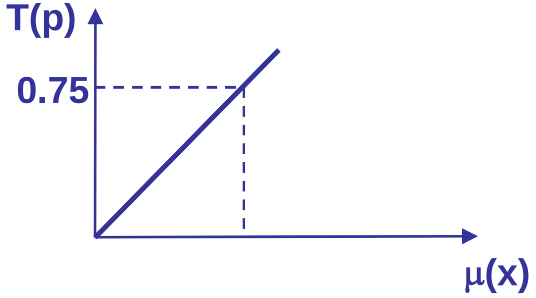
So, the truth value of the proposition P is a fuzzy set defined on [0, 1]

An example

p*: *Temperature* is *high

X **L**





Degrees of truth and probability

Degrees of truth are often confused with **probabilities**.

They are conceptually distinct: fuzzy truth represents membership to vaguely defined sets, not likelihood of some event or condition.

To illustrate the difference, consider this scenario:

> Bob is in a house with two adjacent rooms: the kitchen and the dining room. In many cases, Bob's status within the set of things "in the kitchen" is completely plain: he's either "in the kitchen" or "not in the kitchen". What about when Bob stands in the doorway? He may be considered as "partially in the kitchen". Quantifying this partial state yields a fuzzy set membership. With only his little toe in the dining room, we might say Bob is 0.99 "in the kitchen", for instance. No event (like a coin toss) will resolve Bob to being completely "in the kitchen" or "not in the kitchen", as long as he's standing in that doorway.

Fuzzy modifiers

Fuzzy modifiers modify truth values

x* is *Young means **(*x* is *Young*) is *true***

It can be modified as:

x is *very Young* is *true*

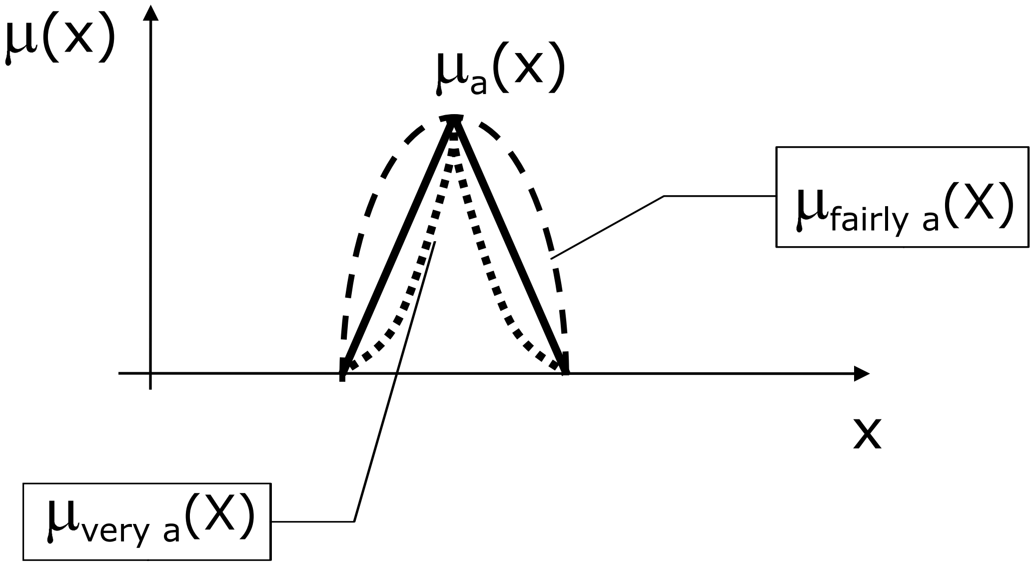
x is *Young* is *very true*

x is *very Young* is *very true*

Examples of fuzzy modifiers

$\mu_{\text{very a}}(x) = \mu_a(x)^2$

$\mu_{\text{fairly a}}(x) = \mu_a(x)^{1/2}$



Kind of modifiers

- **Strong modifiers**

  m(a) <= a for each a in [0, 1]

  They make the predicate stronger, so they reduce the truth of the proposition

- **Weak modifiers**

  m(a) >= a for each a in [0, 1]

  They make the predicate weaker, so they increase the truth of the proposition

Properties of fuzzy modifiers

They have to satisfy these properties:

- $m(0)=0$ and $m(1)=1$
- m is a continuous function
- if m is strong m^{-1} is weak, and the other way round
- Given another modifier g, the composition of g and m and viceversa are modifiers, too, and ,if both are strong (weak), so is their composition

Qualified, non-conditional propositions

p: (X is F) is S

S is a fuzzy truth qualifier

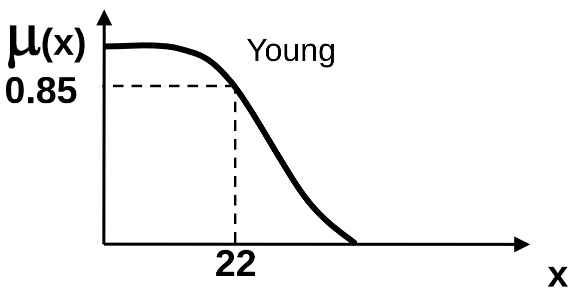
F is a fuzzy set

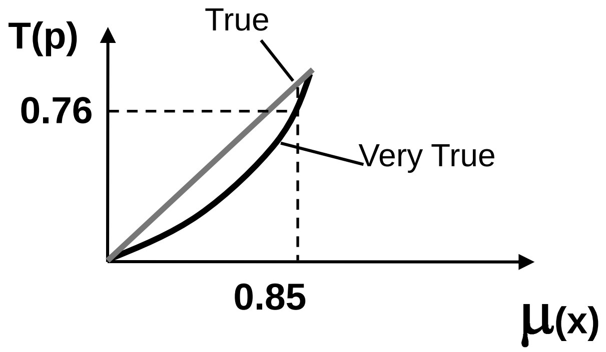
P is *truth qualified*

Example

p*: *Tina* is *Young* is *very true

X F S





What to remember from these slides?

- What is a fuzzy logic
- The conceptual path from fuzzy sets to fuzzy logic
- Fuzzy modifiers