Introduction to fuzzy sets

Andrea Bonarini

Artificial Intelligence and Robotics Lab
Department of Electronics and Information
Politecnico di Milano

E-mail: bonarini@elet.polimi.it
URL:http://www.dei.polimi.it/people/bonarini

A bit of history

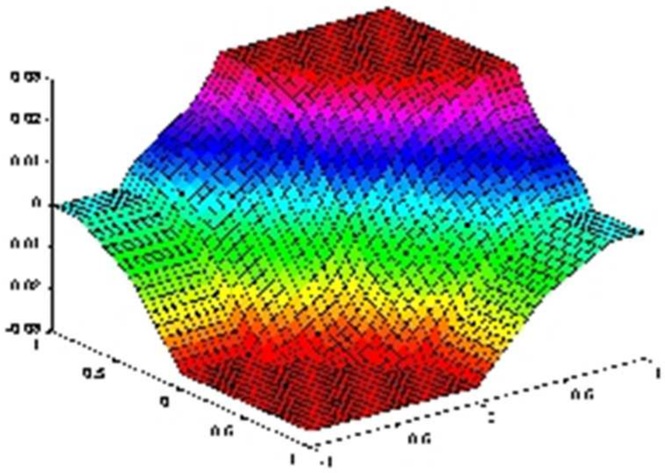
- Fuzzy sets have been defined by Lotfi Zadeh in 1965, as a tool to model approximate concepts
- In 1972 the first “linguistic” fuzzy controller is implemented
- In the Eighties boom of fuzzy controllers first in Japan, then USA and Europe
- In the Nineties applications in many fields: fuzzy data bases, fuzzy decision making, fuzzy clustering, fuzzy learning classifier systems, neuro-fuzzy systems…
  Massive diffusion of fuzzy controllers in end-user goods
- Now, fuzzy systems are the kernel of many “intelligent” devices

Main characteristics

Fuzzy sets:
precise model in a finite number of points, smooth transition (approximation) among them.

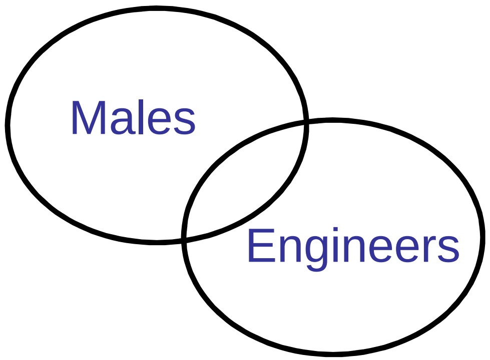
E.g.: **control of a power plant**.

We can define what to do at the regimen (e.g., steam temperature =120°, steam pressure 3 atm), and when in critical situations (e.g., steam temperature= 100°), and design a model that smoothly goes from one point to the other.

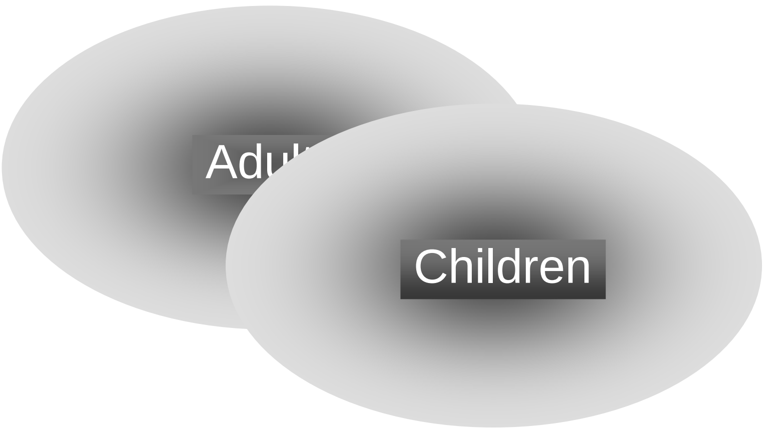


What is a fuzzy set?

A fuzzy set is a set whose membership function may range on the interval [0,1].



Crisp sets



Fuzzy sets

Fuzzy membership functions

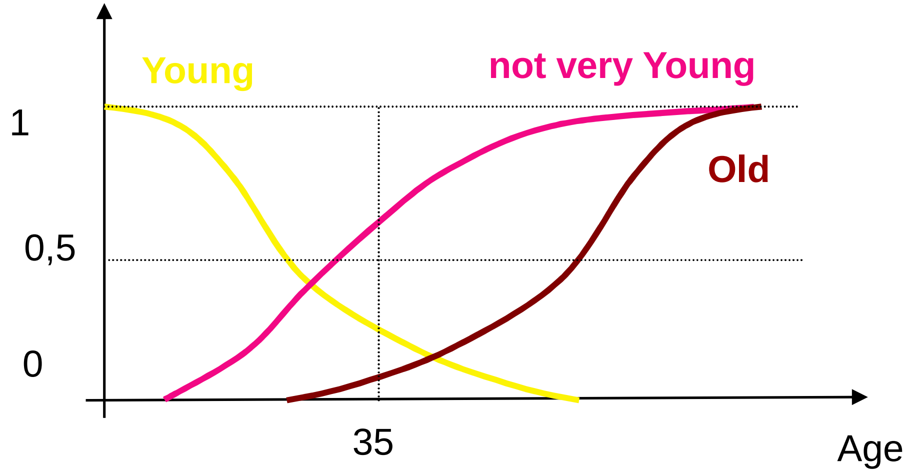
A membership function defines a set

Defines the degree of membership of an element to the set

$$\mu: U \rightarrow [0, 1]$$

A 35 years old person is:

- **Young** with membership 0,3
- **Old** with membership 0,2
- **not very Young** with membership 0,6

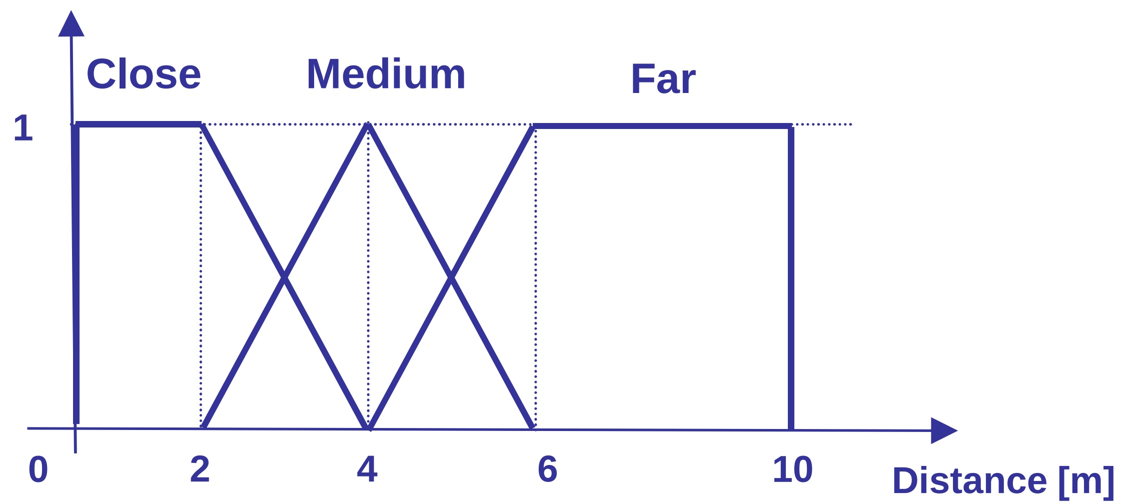


How to define MFs

1. Select a variable
2. Define the range of the variable
3. Identify labels
4. For each label identify characteristic points
5. Identify function shapes
6. Check

Let's try to define some MFs

First of all, the variable…	→	Distance
Range of the variable	→	[0..10]
Labels	→	Close, Medium, Far
Characteristic points	→	0, max, middle values, where MF=1, …
Function shape	→	Linear



MFs and concepts

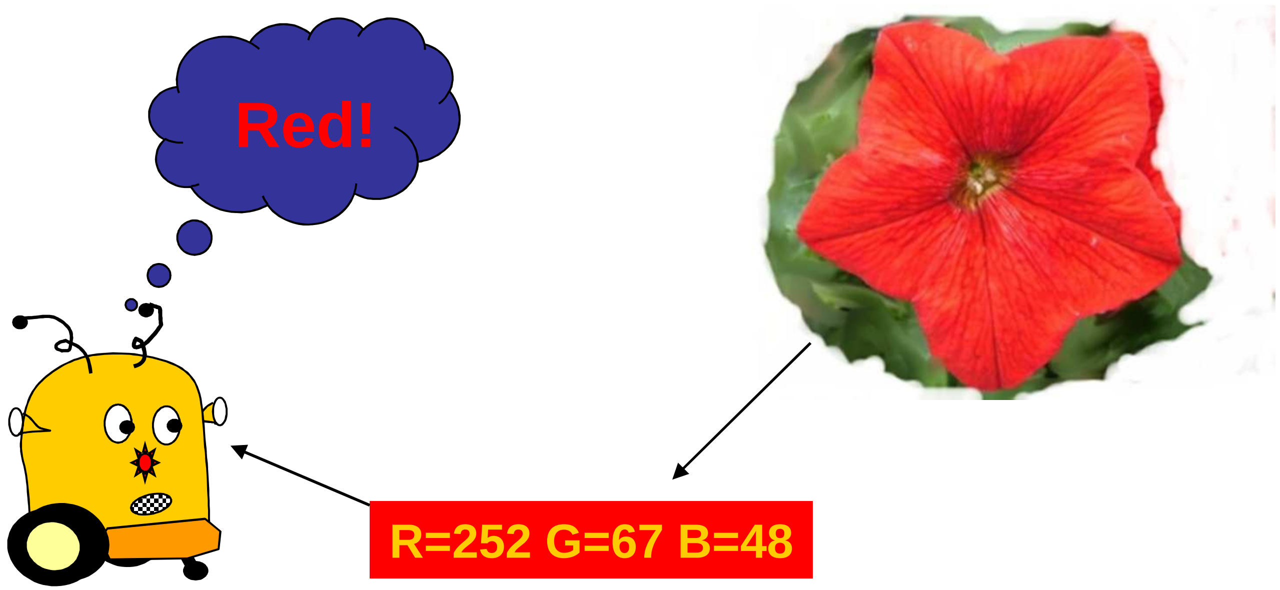
MFs **define** fuzzy sets

Labels **denote** fuzzy sets

Fuzzy sets can be considered as **conceptual** representations

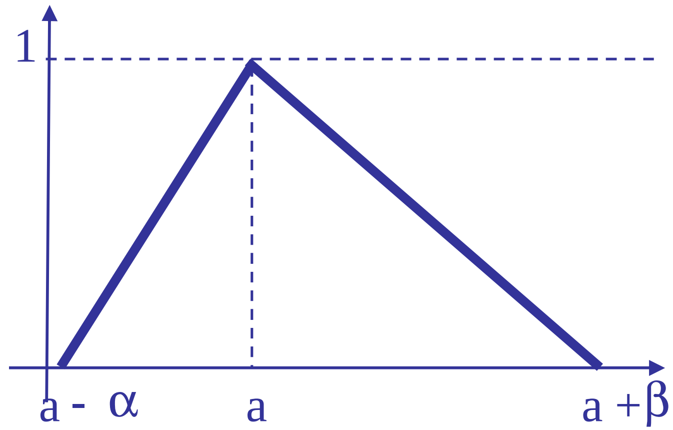
Symbol grounding:

reason in terms of concepts and ground them on objective reality

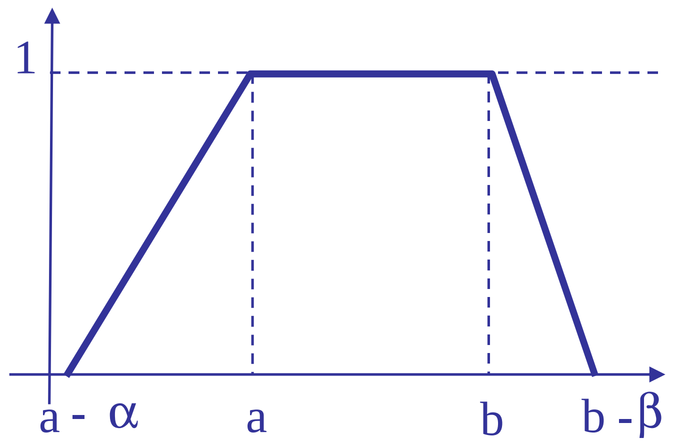


Some conceptual differences

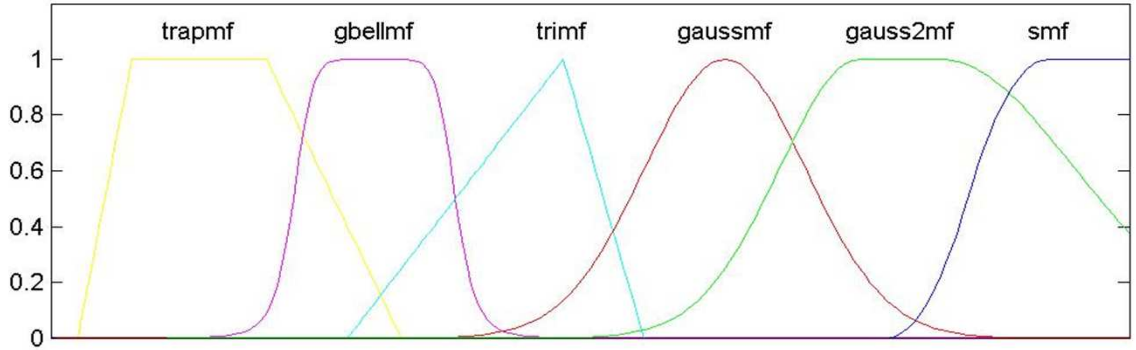
A fuzzy set with only one member with the maximum membership

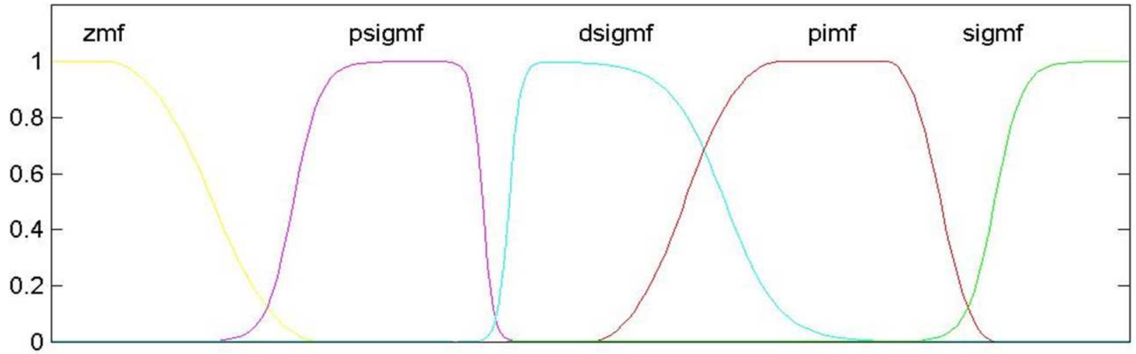


A fuzzy set with a set of members with the maximum membership

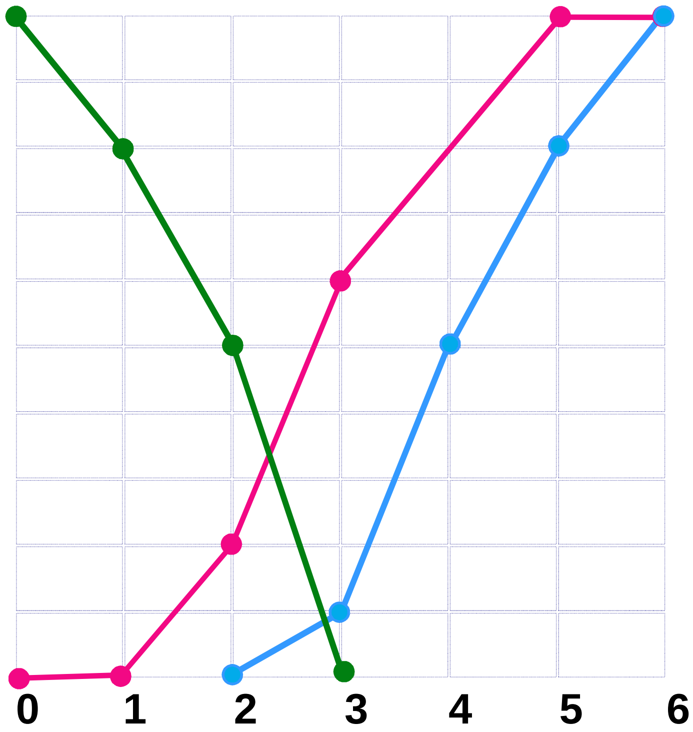


Some variations

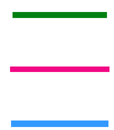




Fuzzy sets on ordinal scales

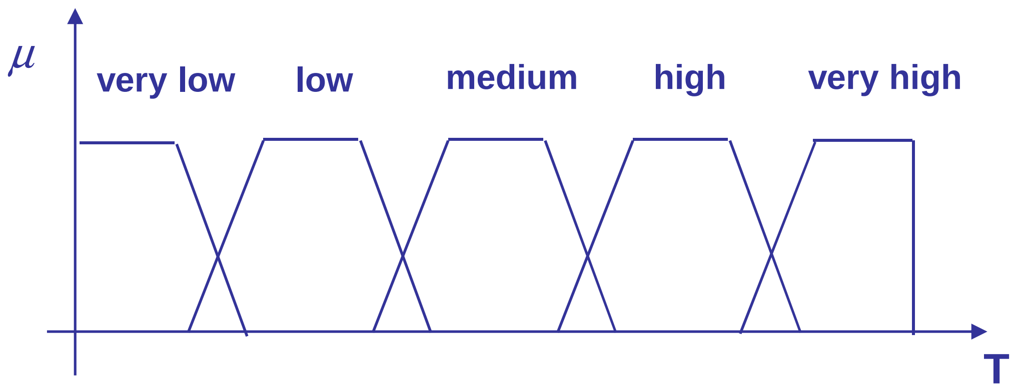


0 - no education
1 - elementary school
2 - high school
3 - two year college
4 - bachelor's degree
5 - masters's degree
6 - doctoral degree



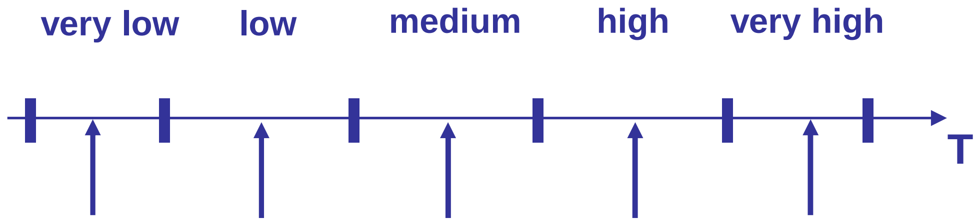
poorly educated
highly educated
very highly educated

Fuzzy sets and intervals



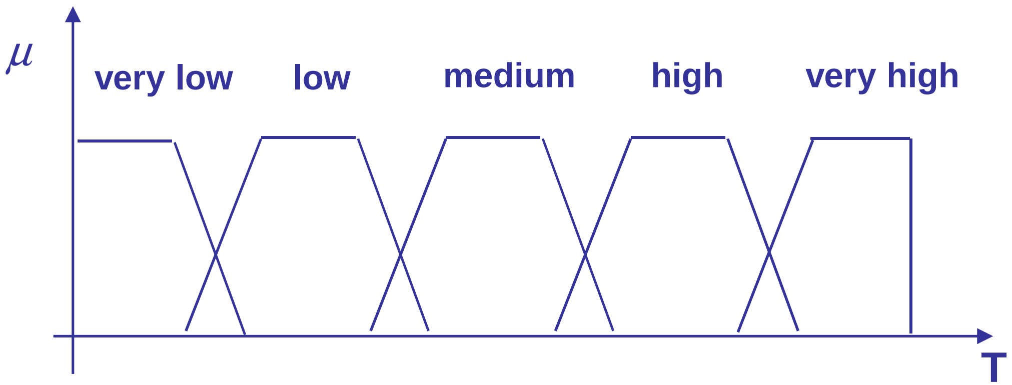
Smoother transition

in labeling a value



Frame of cognition

Fuzzy sets covering the universe of discourse



Each fuzzy set is a *granule*

Properties of a frame of cognition

Coverage

Each element of the universe of discourse is assigned to at least a granule with membership > 0

Unimodality of fuzzy sets

There is a unique set of values for each granule with maximum membership

Fuzzy partition:

for each value of the universe of discourse the sum of membership degrees to the corresponding granules is 1

Robustness

Let's consider a punctual error as the sum of the errors in interpretation by fuzzy sets due to imprecise measurements, noise, …

$$e(\hat{a}) = |\mu_1(\hat{a}) - \mu_1(a')| + ... + |\mu_n(\hat{a}) - \mu_n(a')|$$

and the integral error, as the integral of e(a) over the range of a

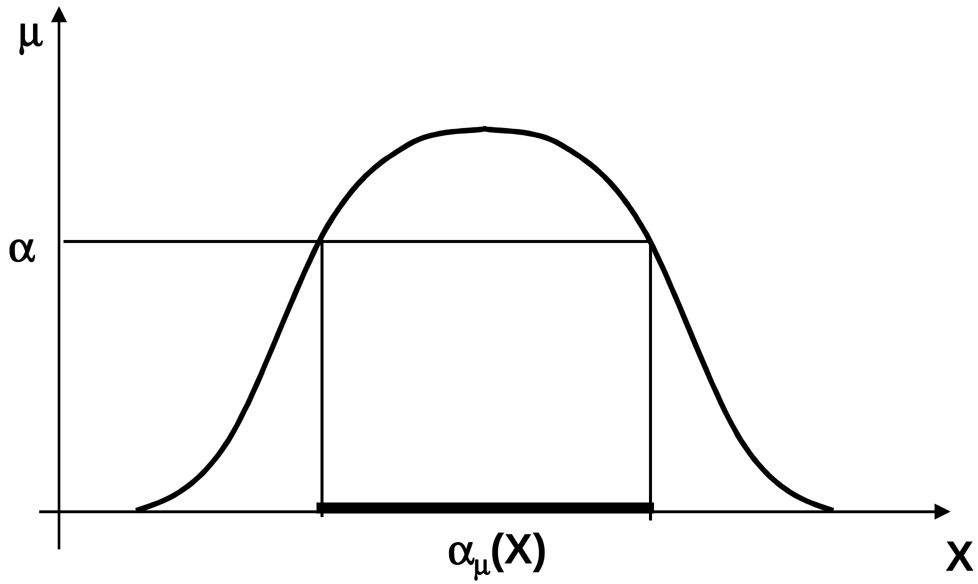
$$e_i = \int e(a)\, da$$

It can be demonstrated that the integral error of a fuzzy partition is smaller than that of a boolean partition, and that it is minimum w.r.t. any other frame of cognition.

α-cuts

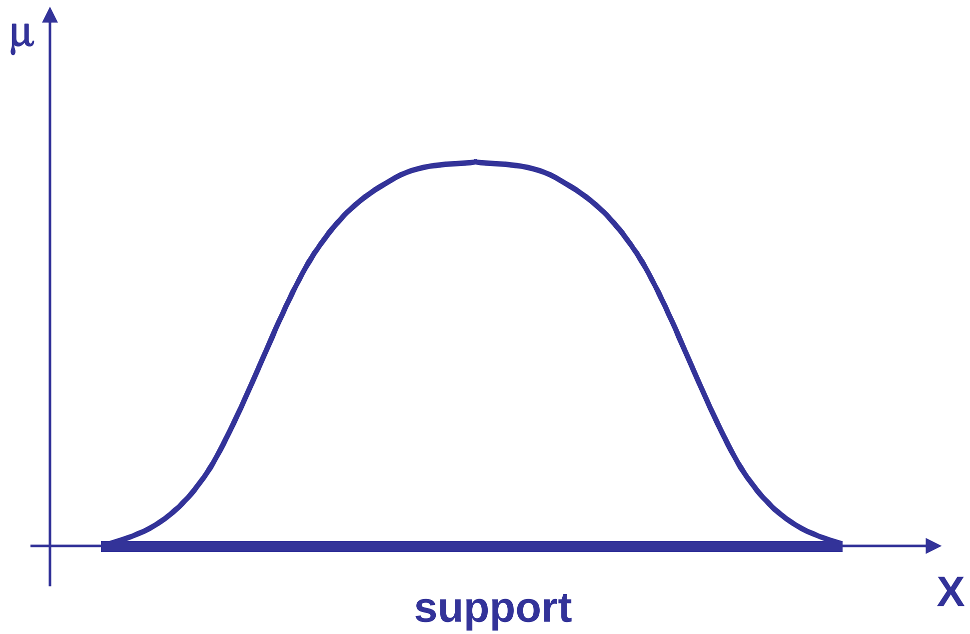
The α-cut of a fuzzy set is the crisp set of the values of x such that $\mu(\mathbf{X}) \geq \alpha$

$$\alpha_\mu(X) = \{x \mid \mu(x) \geq \alpha\}$$



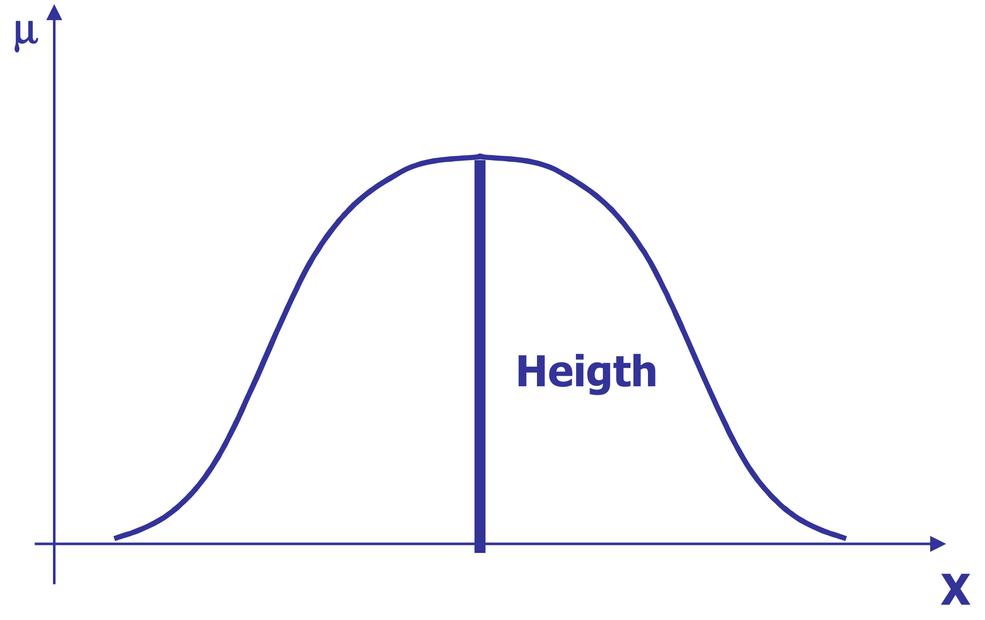
Support of a fuzzy set

The crisp set of values x of X such that $\mu_f(x) > 0$ is the support of the fuzzy set f on the universe X



Height of a fuzzy set

The height *h(A)* of a fuzzy set A on the universe X is the highest membership degree of an element of X to the fuzzy set



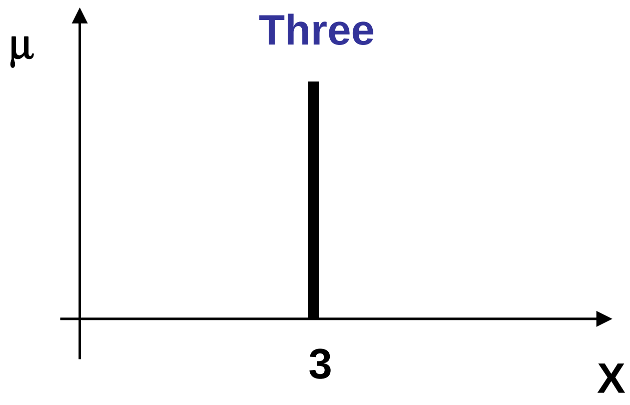
A fuzzy set f is *normal* iff $h_f(x)=1$

“Strange” MFs

Singleton:

a fuzzy set with

one member

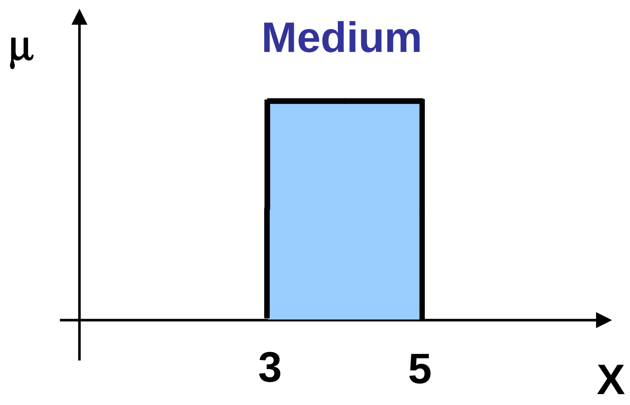


Interval:

a fuzzy set with

members all at

the maximum height

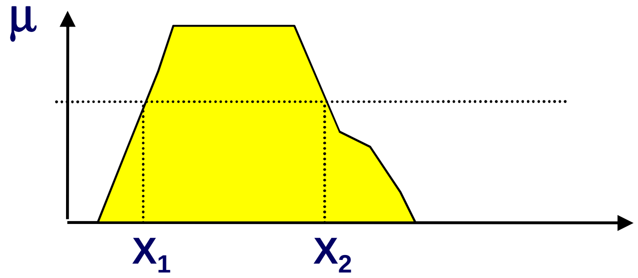


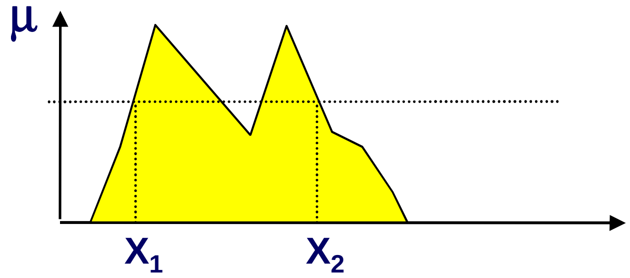
Convex fuzzy sets

A fuzzy set is *convex* iff

$$\mu(\lambda x_1 + (1-\lambda) x_2) \geq \min [\mu(x_1), \mu(x_2)]$$

for any x_1, x_2 in $\Re$ and any λ belonging to [0,1]





Standard operators on fuzzy sets

Complement

$$\mu_{\bar{f}}(x) = 1 - \mu_f(x)$$

Union

$$\mu_{f1 \cup f2}(x) = \max\ [\mu_{f1}(x), \mu_{f2}(x)]$$

Intersection

$$\mu_{f1 \cap f2}(x) = \min\ [\mu_{f1}(x), \mu_{f2}(x)]$$

Standard operators on fuzzy sets

Complement

$$\mu_{\neg f}(x)=1-\mu_f(x)$$

Union

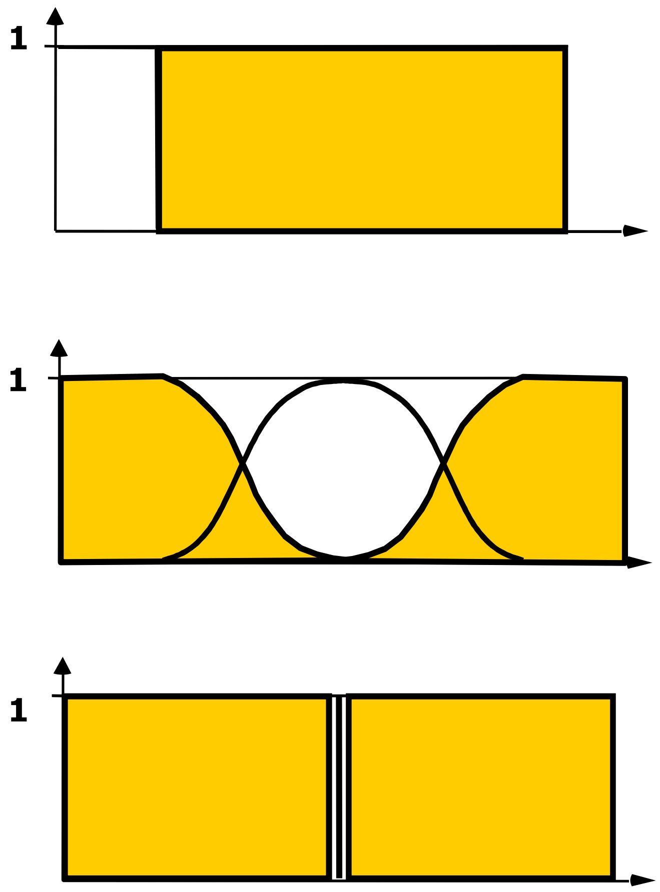
$$\mu_{f_1 \cup f_2}(x) = \max(\mu_{f_1}(x), \mu_{f_2}(x))$$

Intersection

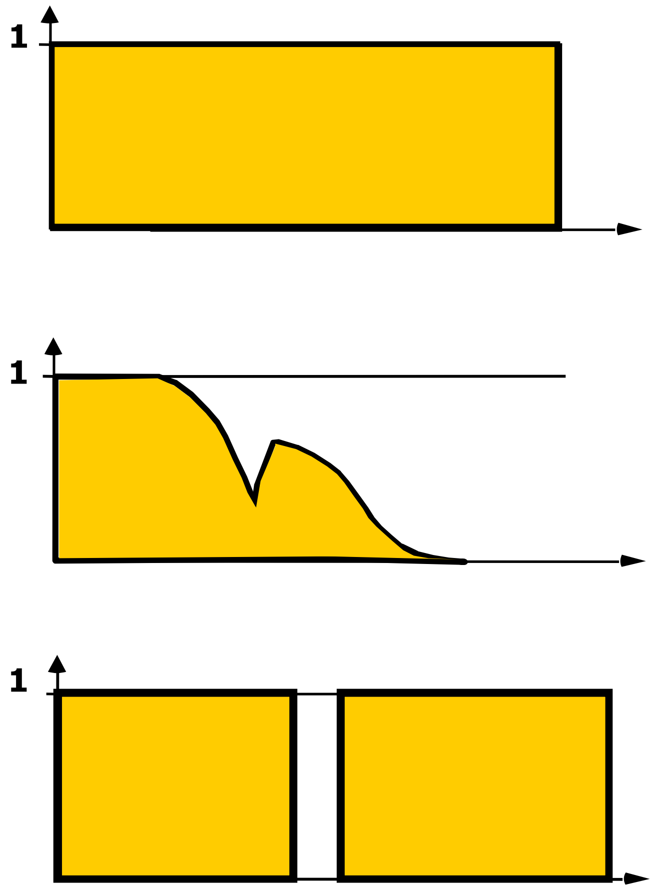
$$\mu_{f_1 \cap f_2}(x) = \min(\mu_{f_1}(x), \mu_{f_2}(x))$$

Examples of operator application

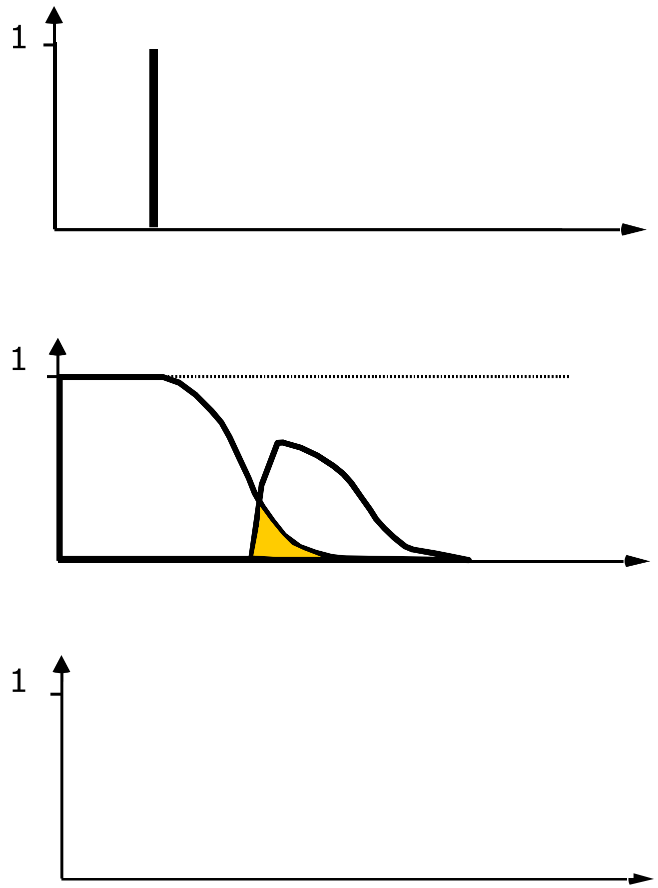
Complement



Union

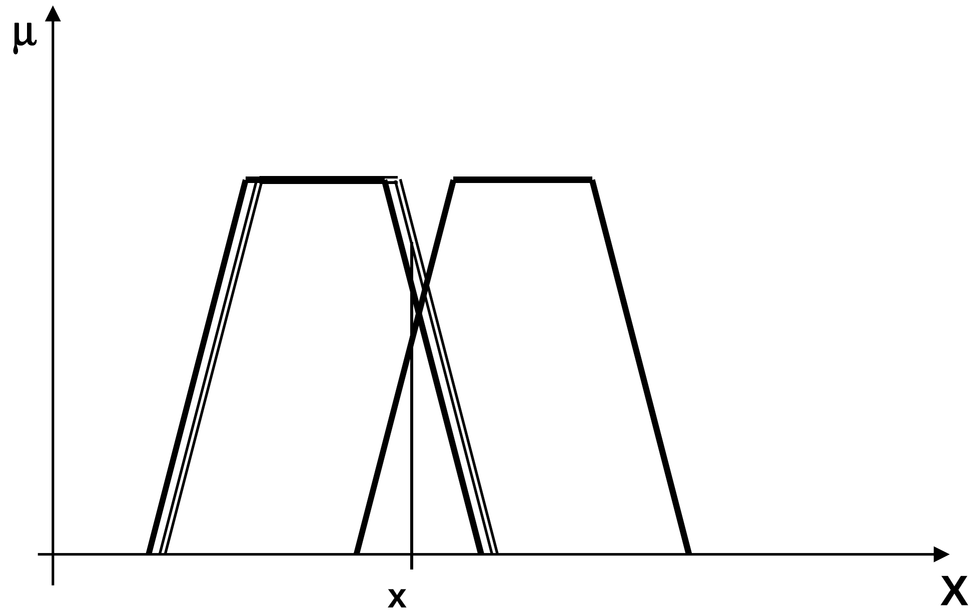


Intersection



Fundamental property of standard operators

Using the standard operators the maximum error is the one we have on the operand's MFs



Complement

$$c : [0,1] \to [0,1]$$

$$c(\mu_A(x)) = \mu_{\neg A}(x)$$

Axioms:

1. c(0)=1; c(1)=0 (*boundary conditions*)
2. For all a and b in [0,1], if $a < b$ then $c(a) \geq c(b)$ (*monotonicity*)
3. c is a *continuous function*
4. c is *involutive*, i.e., c(c(a))=a for all a in [0,1]

Intersection and T-norms

$$\mu_{A\cap B}(x) = i[\mu_A(x), \mu_B(x)]$$

Axioms:

1. $i[a, 1]=a$ (*boundary conditions*)
2. $d \geq b$ implies $i(a,d) \geq i(a,b)$ (*monotonicity*)
3. $i(b,a) = i(a,b)$ (*commutativity*)
4. $i(i(a,b),d) = i(a,i(b,d))$ (*associativity*)
5. i is continuous
6. $a \geq i(a,a)$ (*sub-idempotency*)
7. $a_1 < a_2$ and $b_1 < b_2$ implies that $i(a_1,b_1) < i(a_2,b_2)$ (*strict monotonicity*)

T-norms: examples

$$\frac{ab}{\max[\ a, b, \alpha\ ]}$$

for α=1 we have ***ab***

for α=0 we have ***min(a, b)***

$$t_1(\mu_A(x), \mu_B(x)) = \max(0, \mu_A(x) + \mu_B(x) - 1)$$

$$t_{2.5}(\mu_A(x), \mu_B(x)) = \frac{\mu_A(x) \cdot \mu_B(x)}{\mu_A(x) + \mu_B(x) - \mu_A(x) \cdot \mu_B(x)}$$

Union and T-conorms (S-norms)

$$\mu_{A \cup B}(x) = u[\mu_A(x), \mu_B(x)]$$

Axioms:

1. $u[a, 0]=a$ (*boundary conditions*)
2. $b \leq d$ implies $u(a,b) \leq u(a,d)$ (*monotonicity*)
3. $u(a,b) = u(b,a)$ (*commutativity*)
4. $u(a,u(b,d)) = u(u(a,b),d)$ (*associativity*)
5. u is continuous
6. $u(a,a) \geq a$ (*super-idempotency*)
7. $a_1 < a_2$ e $b_1 < b_2$ implies that $u(a_1,b_1)<u(a_2,b_2)$ (strict *monotonicity*)

T-conorms: examples

$$s(\mu_A(x), \mu_B(x)) = \min\{1, (\mu_A(x)^p + \mu_B(x)^p)^{1/p} \quad p \geq 1$$

$$s_1(\mu_A(x), \mu_B(x)) = \min(1, \mu_A(x) + \mu_B(x))$$

$$s_3(\mu_A(x), \mu_B(x)) = \max(\mu_A(x), \mu_B(x))$$

$$s_+(\mu_A(x), \mu_B(x)) = \mu_A(x) + \mu_B(x) - \mu_A(x) * \mu_B(x)$$

Aggregation

$$\mu_A(x) = h[\mu_{A1}(x), \ldots, \mu_{An}(x)]$$

Axioms:

1. h[0,..., 0]=0, h[1,..., 1]=1 (*boundary conditions*)
2. *monotonicity*
3. h is continuous
4. h(a,..,a) = a (*idempotency*)
5. *simmetricity*

Properties of aggregation

$\min(a_1, \ldots, a_n) \le h(a_1, \ldots, a_n) \le \max(a_1, \ldots, a_n)$

Example of aggregation operator: generalized average

$h(a_1, \ldots, a_n) = (a_1^{\alpha} + \ldots + a_n^{\alpha})^{1/\alpha} / n$