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Unmanned autonomous vehicles in air land and sea

Land vehicle kinematic models

Luca Bascetta
luca.bascetta@polimi.it

MEchatronics & Robotics Laboratory for Innovation - Politecnico di Milano

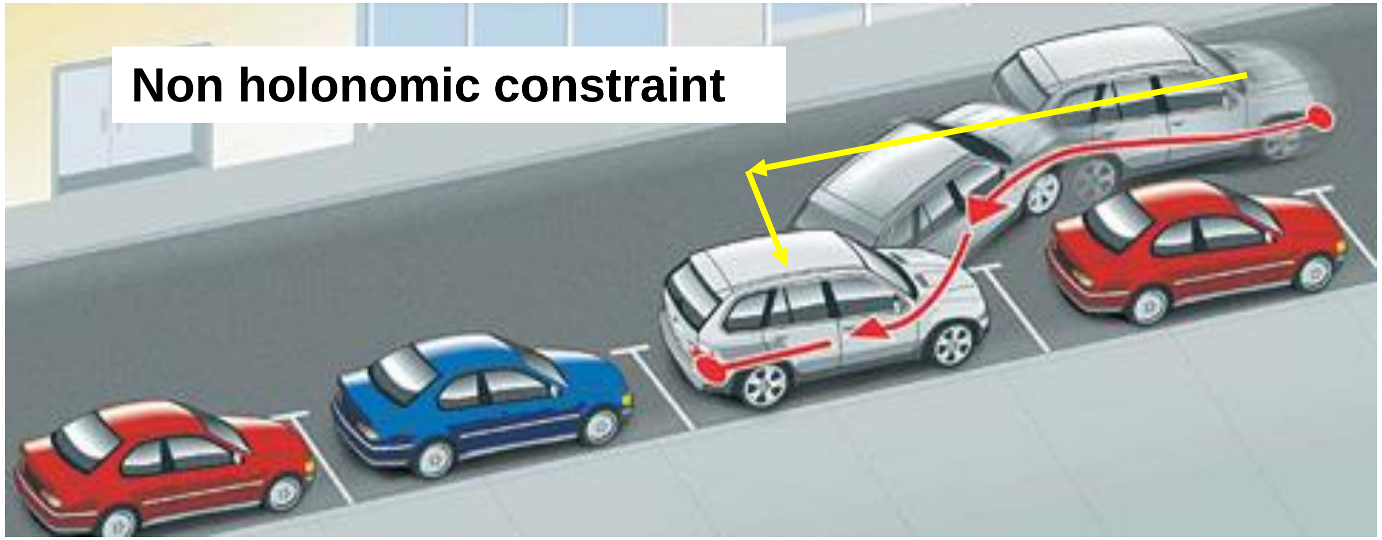
Why modeling?

Modeling of a control system is important for several reasons

- Models are the base of the deliberative paradigm, e.g., for task planning or trajectory planning
- Models are the base of some reactive approaches as well
  - Model based control
  - Model predictive control
- Models are used for sensing and perception
- Models are used for simulations

We will see most of these except simulation …

Motion constrains



The motion of the vehicle is represented by a set of generalized coordinates $\mathbf{q} \in \mathbb{R}^n$

We define holonomic constraint each constraint

$$h_i(\mathbf{q}) = 0 \qquad i = 1, \ldots, k < n$$

These constraints reduce the vehicle configuration space

Kinematic constraints

Kinematic constraints limit the set of generalized velocities for each configuration

$$a_i\left(\mathbf{q},\dot{\mathbf{q}}\right)=0 \qquad i=1,\ldots,k<n$$

they are usually linear with respect to generalized velocities

$$\mathbf{a}_i^T\left(\mathbf{q}\right)\dot{\mathbf{q}}=0 \qquad i=1,\ldots,k<n$$

$$\mathbf{A}^T\left(\mathbf{q}\right)\dot{\mathbf{q}}=\mathbf{0}$$

If a system is characterized by *k* holonomic constraints, we define kinematic constraints

$$\frac{dh_i\left(\mathbf{q}\right)}{dt}=\frac{\partial h_i\left(\mathbf{q}\right)}{\partial\mathbf{q}}\dot{\mathbf{q}}=0 \qquad i=1,\ldots,k$$

We can now summarize as: $\mathbf{h}\left(\mathbf{q}\right)=\mathbf{0} \implies \mathbf{A}^T\left(\mathbf{q}\right)\dot{\mathbf{q}}=\mathbf{0}$

$$\mathbf{A}^T\left(\mathbf{q}\right)\dot{\mathbf{q}}=\mathbf{0} \not\Longrightarrow \mathbf{h}\left(\mathbf{q}\right)=\mathbf{0}$$

 k nonholonomic constraints

Holonomic vs nonholonomic constraints

Consider one kinematic constraint

$$\mathbf{a}^T(\mathbf{q})\,\dot{\mathbf{q}} = 0$$

Assuming it is a holonomic constraint, we integrate it obtaining

$$h(\mathbf{q}) = c$$

$$\frac{\partial h}{\partial \mathbf{q}} = \gamma(\mathbf{q})\,\mathbf{a}^T(\mathbf{q})$$

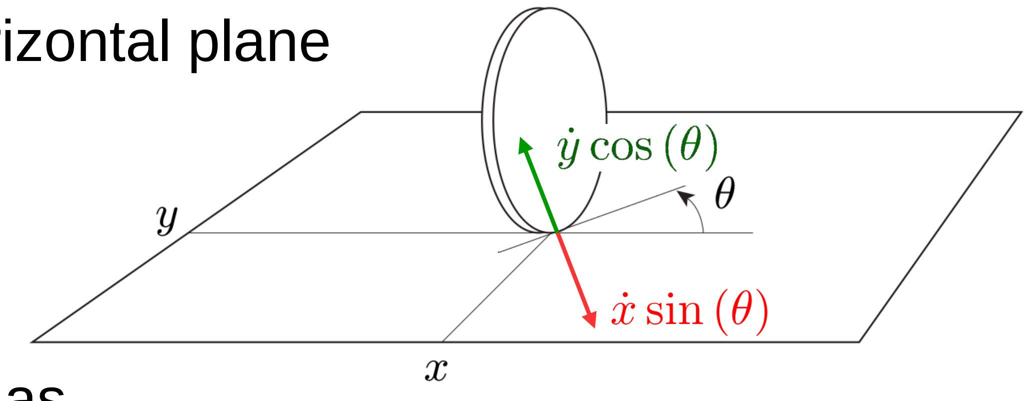
The motion of the system is limited by the constraint to a level surface

Assume now the constraint is non holonomic

- There is no reduction in the number if generalized coordinates
- There is a reduction in the number of degrees of freedom given by $\dot{\mathbf{q}} \in \mathcal{N}\left(\mathbf{a}^T(\mathbf{q})\right)$

Example: rolling disk

Consider a disc that rolls without sliding on a horizontal plane



- The disc configuration is described by three generalized coordinates

$$\mathbf{q} = \begin{bmatrix} x & y & \theta \end{bmatrix}^T$$

- The pure rolling constraint can be written as

$$\dot{x}\sin(\theta) - \dot{y}\cos(\theta) = \begin{bmatrix} \sin(\theta) & \cos(\theta) & 0 \end{bmatrix} \dot{\mathbf{q}} = 0$$

- This is a non holonomic constraint, as it does not reduced the number of generalized coordinates
- The line passing through the wheel contact point and having the direction orthogonal to the sagittal axis of the vehicle is called <u>zero motion line</u>

Using constraints to derive kinematic models

Given a set of k kinematic constraints, the generalized velocities are constrained, for every configuration, to be on the null space of the constraint matrix

Consider a base $\{\mathbf{g}_1(\mathbf{q}), \ldots, \mathbf{g}_{n-k}(\mathbf{q})\}$ of the matrix $\mathcal{N}\left(\mathbf{A}^T(\mathbf{q})\right)$

The admissible trajectories of the mechanical system are thus the solutions of the following nonlinear dynamical system (kinematic model)

$$\dot{\mathbf{q}} = \sum_{j=1}^{m} \mathbf{g}_j(\mathbf{q})\, u_j = \mathbf{G}(\mathbf{q})\,\mathbf{u} \qquad m = n - k$$

The choice of the base ***g*** is not unique, the components of the input vector ***u*** may thus have different physical interpretations

Holonomicity and controllability

Holonomicity and non holonomicity are related to controllability of the kinematic model

- A kinematic model is controllable if, given two arbitrarily selected configurations $\boldsymbol{q}_i$ and $\boldsymbol{q}_f$, an input vector $\boldsymbol{u}$ exists generating a trajectory that takes the system from $\boldsymbol{q}_i$ to $\boldsymbol{q}_f$ without violating the kinematic constraints

- If the model is controllable, all the kinematic constraints are nonholonomic, while If the model is not completely controllable, some of the kinematic constraints are holonomic

Let's make sense of it with a few examples!

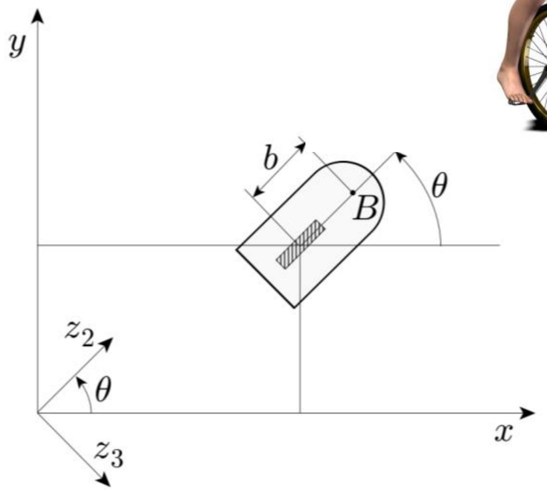
Unicycle kinematic model

The unicycle is described by three generalized coordinates

$$\mathbf{q} = \begin{bmatrix} x & y & \theta \end{bmatrix}^T$$

Assuming a pure rolling wheel we can introduce the following nonholonomic constraint

$$\dot{x}\sin(\theta) - \dot{y}\cos(\theta) = \begin{bmatrix} \sin(\theta) & \cos(\theta) & 0 \end{bmatrix} \dot{\mathbf{q}} = 0$$



A base of the null space of the constraint matrix is given by

$$\mathbf{G}(\mathbf{q}) = \begin{bmatrix} \mathbf{g}_1(\mathbf{q}) & \mathbf{g}_2(\mathbf{q}) \end{bmatrix} = \begin{bmatrix} \cos(\theta) & 0 \\ \sin(\theta) & 0 \\ 0 & 1 \end{bmatrix}$$

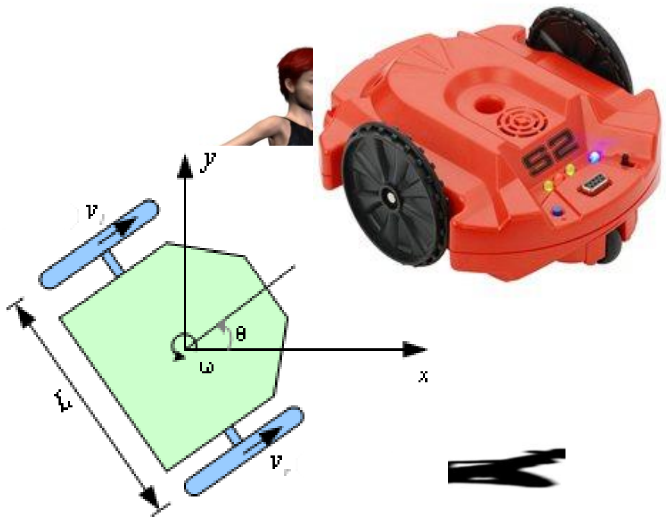
All the admissible generalized velocities at a given configuration ***q*** are given by the linear combination of $\boldsymbol{g_1(q)}$ and $\boldsymbol{g_2(q)}$

Unicycle kinematic model and

Given the previous $g_1(q)$ and $g_2(q)$ the Unicycle kinematic model is thus given by

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} \cos(\theta) \\ \sin(\theta) \\ 0 \end{bmatrix} v + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \omega$$

where v and ω are the linear and angular velocities of the vehicle



A unicycle is a vehicle with a serious balancing problem, especially in static conditions, more stable vehicles, kinematically equivalent, are differential-drive robots.

For differential-drive robots the inputs are the wheel rotational velocities ω_R and ω_L which are related to the unicycle model inputs by

$$v = \frac{r\left(\omega_R + \omega_L\right)}{2} \qquad \omega = \frac{r\left(\omega_R - \omega_L\right)}{L}$$

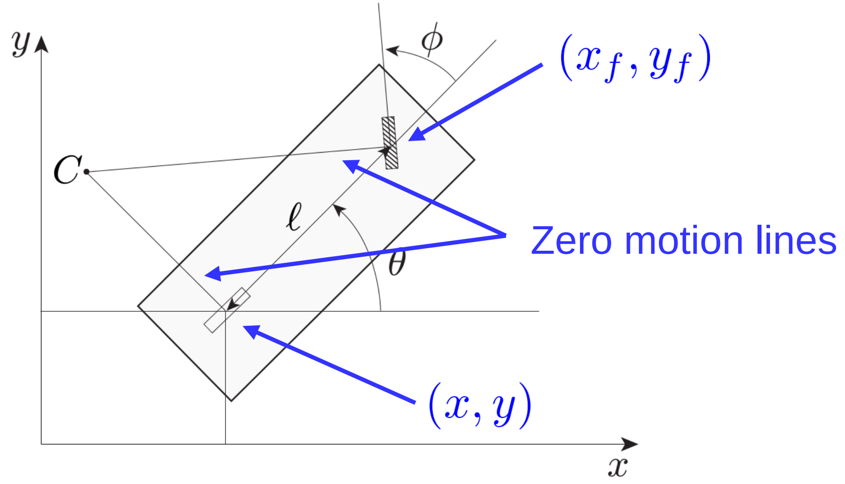
Bycicle kinematic model

The bicycle is described by four generalized coordinates

$$\mathbf{q} = \begin{bmatrix} x & y & \theta & \phi \end{bmatrix}^T$$

Assuming pure rolling wheels, we can introduce the following nonholonomic constraints

$$\dot{x}_f \sin(\theta + \phi) - \dot{y}_f \cos(\theta + \phi) = 0$$
$$\dot{x} \sin(\theta) - \dot{y} \cos(\theta) = 0$$



The two zero motion lines intersect in point C, called instantaneous center of rotation (ICR) and its position depends only on the bicycle configuration

Bycicle kinematic model

Introducing the rigid body constraint

$$x_f = x + \ell \cos(\theta) \qquad y_f = y + \ell \sin(\theta)$$

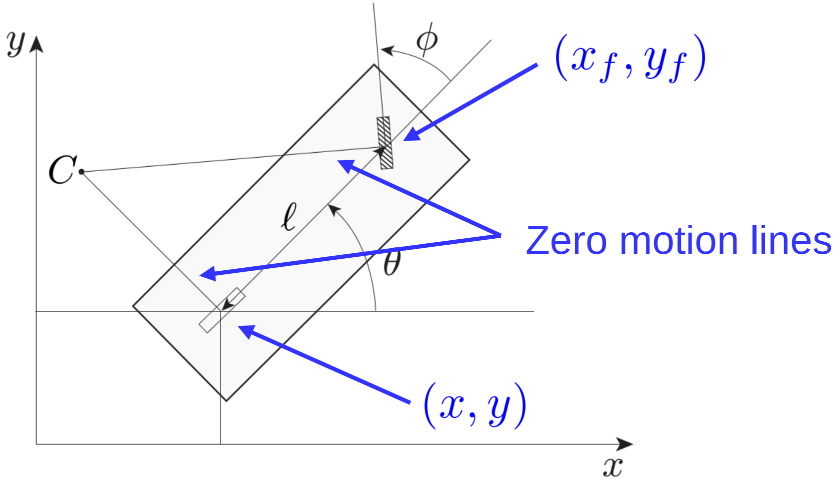
We can rewrite the first constraint in terms of the generalized coordinates

$$\dot{x} \sin(\theta + \phi) - \dot{y} \cos(\theta + \phi) - \ell \dot{\theta} \cos(\phi) = 0$$

The constraint matrix is thus

$$\mathbf{A}^T(\mathbf{q}) = \begin{bmatrix} \sin(\theta) & -\cos(\theta) & 0 & 0 \\ \sin(\theta+\phi) & -\cos(\theta+\phi) & -\ell\cos(\phi) & 0 \end{bmatrix}$$

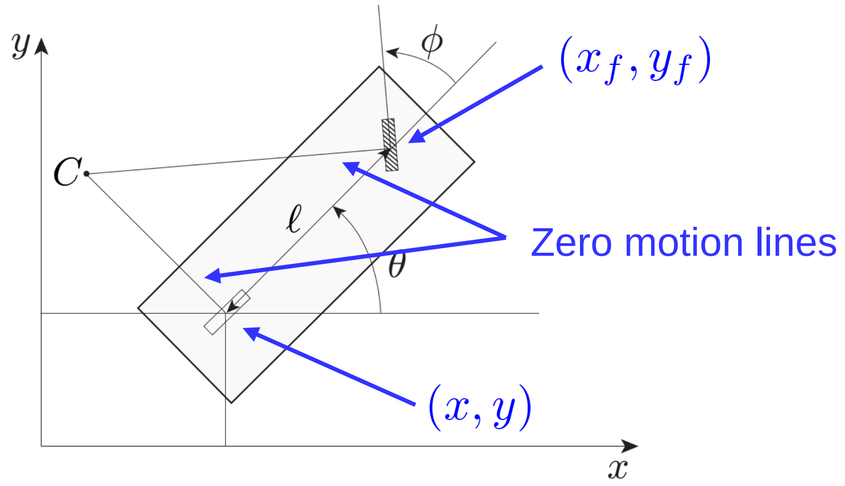
Having rank equal to 2 the null space has dimension n-k=2



Bycicle kinematic model

All the admissible generalized velocities at a given configuration ***q*** are given by

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \\ \dot{\phi} \end{bmatrix} = \begin{bmatrix} \cos(\theta)\cos(\phi) \\ \sin(\theta)\cos(\phi) \\ \sin(\phi)/\ell \\ 0 \end{bmatrix} u_1 + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} u_2$$



Since the front wheel orientation can be changed we have

$$u_2 = \omega$$

where ω is the steering velocity.

In case of *front-wheel drive* we have

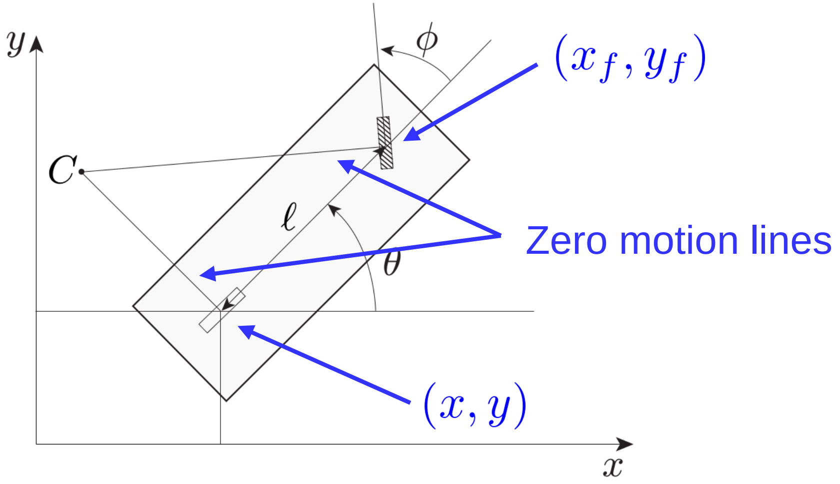
$$u_1 = v$$

Bycicle kinematic model

In case of *rear wheel drive* the first two equation should be equivalent to a unicycle then we have

$$u_1 = \frac{v}{\cos(\phi)} \qquad u_2 = \omega$$

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \\ \dot{\phi} \end{bmatrix} = \begin{bmatrix} \cos(\theta) \\ \sin(\theta) \\ \tan(\phi)/\ell \\ 0 \end{bmatrix} v + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \omega$$



As for the unicycle, the bicycle has serious balancing problems, especially in static conditions. More stable but kinematically equivalent vehicles are tricycle and car-like robots

Wrap-up slide on “Land vehicle kinematic models”

What should remain from this lecture?

- Difference between holonomic and non holonomic constraints
- The derivation of kinematics as null space of the kinematics constraints
- The kinematic model of the unicycle (a.k.a. Differential drive model)
- The kinematic model of the bycicle (a.k.a. Single track model)

References

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